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Algebraic Structure of Symbolic Expressions

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ABSTRACT

The ring S_∞ of formal power series in the noncommuting variables $a_1, \ldots, a_n$ with the coefficient field GF(2) is introduced and studied. The term *symbolic expression* is used instead of formal power series, since it generalizes the concept of a symbolic expression introduced in [2] and [3]. S_∞ is characterized as the terminal object of the category **Aut** of automata. A category theoretic characterization of the subring S^{rat} of S_∞ consisting of rational sexps is also given.

Introduction

Theory of formal power series in noncommuting variables provides a useful algebraic tool for the study of formal languages. In the most general setting of the theory, the coefficients of a formal power series are taken from an arbitrary *semiring*; and it is possible to prove useful theorems in this general setting. (See e.g. Salomaa and Soittola[1].) In this paper, however, we will take the two elemented Galois field GF(2) as the coefficient semiring. This choice of the semiring will turn out to be very convenient. We will use the term *symbolic expression* (or *sexp* for short) instead of formal power series, since it generalizes the concept of a symbolic expression introduced in Sato[2] and Sato and Hagiya[3]. In [2] and [3], it was shown that symbolic expressions constitute a flexible data structure; and a programming language called Hyperlisp which computes (partial) recursive functions of symbolic expressions was introduced. Here we study symbolic expressions from an algebraic point of view.

In 1, we study the ring S_∞ consisting of all the sexps. We show that S_∞ satisfies a certain domain equation for an abstract data structure.

In 2, we characterize S_∞ as the terminal object of the category **Aut** of automata. We then study the subring S^{rat} of S_∞ consisting of *rational* sexps. The well-known characterization of rational sexps in terms of finite automata is established in our formalism. A category theoretic characterization of S^{rat} is also given.

In 3, we introduce the subring S of S_∞ consisting of *finite* sexps. The relationship with the concept of a symbolic expression introduced in [2], [3] is also discussed.

1. S_∞

Let $\Sigma=\{a_1, ..., a_n\}$ $(n \geqq 1)$ be an alphabet consisting of n distinct symbols, which we will fix for the rest of this paper. Let $W=\Sigma^*$ be the free monoid over the alphabet Σ, and let $\mathbf{2}=\{0,1\}$ be the Galois field GF(2). We put

$$S_\infty=\{r \mid r: W \to \mathbf{2}\}.$$

We will use r, s, t etc. to denote elements of S_∞ and u, v, w etc. to denote elements of W. Elements of S_∞ are called *symbotic expressions* or *sexps* for short. Elements of W are called *words*. For a word w, $|w|$ denotes its length. We write (r, w) for $r(w)$. We remark that S_∞ may be identified with 2^W (the power set of W) by the correspondence:

$$r \leftrightarrow \mathrm{supp}(r)=\{w \in W \mid (r, w)=1\}.$$

Any sexp r, then, naturally becomes a language over W.

We now define addition and multiplication on S_∞ as follows.

$$(r+s, w)=(r, w)+(s, w),$$
$$(rs, w)=\sum_{w=uv} (r, u)(s, v).$$

Under these operations, S_∞ becomes a noncommutative ring with the 0 and 1 defined by:

$$(0, w)=0,$$
$$(1, w)=\begin{cases} 1 & \text{if } w=1 \text{ (the unit of } W\text{)}, \\ 0 & \text{otherwise.} \end{cases}$$

By identifying $0, 1 \in S_\infty$ with those in $\mathbf{2}$, we assume that $\mathbf{2} \subseteq S_\infty$. We will also regard S_∞ as a vector space over $\mathbf{2}$. Next, for any $w \in W$ we define $\bar{w} \in S_\infty$ by:

$$(\bar{w}, u)=\begin{cases} 1 & \text{if } w=u, \\ 0 & \text{otherwise.} \end{cases}$$

Since the map: $w \to \bar{w}$ is one-to-one and preserves multiplication on W, we will identify $\bar{w}$ with w and assume that $W \subseteq S_\infty$.

Consider the map $\pi: S_\infty \to S_\infty$ defined by

$$\pi(r)=(r, 1).$$

It is a ring homomorphism and satisfies $\pi^2=\pi$. If we regard S_∞ as a vector space, π becomes a projection and we have the direct sum decomposition:

$$S_\infty=\mathrm{Im}\,\pi \oplus \mathrm{Ker}\,\pi.$$

Since $\mathrm{Im}\,\pi=\mathbf{2}$, if we put

$$\mathbf{M}_\infty=\mathrm{Ker}\,\pi=\{r \mid (r, 1)=0\}$$

we have

$$\mathbf{S}_\infty = 2 \oplus \mathbf{M}_\infty \tag{1.1}$$

We put $\mathbf{A}_\infty = \mathbf{S}_\infty - \mathbf{M}_\infty$. Elements of $\mathbf{M}_\infty$ are called *molecules*, and elements of $\mathbf{A}_\infty$ are called *atoms*.

Next, we define a map $\delta: \mathbf{S}_\infty \times W \to \mathbf{S}_\infty$ by:

$$(\delta(r, u), v) = (r, uv).$$

It is an action of the monoid W on $\mathbf{S}_\infty$, since we have

$$\delta(r, 1) = r,$$
$$\delta(\delta(r, u), v) = \delta(r, uv).$$

For a fixed w,

$$\delta(-, w): \mathbf{S}_\infty \to \mathbf{S}_\infty$$

is a linear transformation. In particular, for each i $(1 \leqq i \leqq n)$, we define $\sigma_i: \mathbf{S}_\infty \to \mathbf{S}_\infty$ by

$$\sigma_i(r) = \delta(r, a_i).$$

We have the following

Proposition 1.1. $\sigma_i(st) = \pi(s)\sigma_i(t) + \sigma_i(s)t$ $(1 \leqq i \leqq n)$.

Proof.

$$\begin{aligned}
(\sigma_i(st), w) &= (\delta(st, a_i), w) \\
&= (st, a_i w) \\
&= \sum_{uv = a_i w} (s, u)(t, v) \\
&= (s, 1)(t, a_i w) + \sum_{a_i uv = a_i w} (s, a_i u)(t, v) \\
&= \pi(s)(\sigma_i(t), w) + \sum_{uv = w} (\sigma_i(s), u)(t, v) \\
&= (\pi(s)\sigma_i(t) + \sigma_i(s)t, w). \quad \square
\end{aligned}$$

Note that, by a simple computation, we have $\sigma_i(a_j) = \delta_{ij}$ $(1 \leqq i, j \leqq n)$, where δ_{ij} is Kronecker's delta.

We now regard $\mathbf{S}_\infty$ as a right $\mathbf{S}_\infty$-module. $\mathbf{M}_\infty$ then becomes its submodule (or, in other words, a right ideal of the ring $\mathbf{S}_\infty$).

Theorem 1.2. *$< a_1, \ldots, a_n >$ forms a basis of $\mathbf{M}_\infty$.*

Proof. It suffices to prove the following (a) and (b).

(a) If $r \in \mathbf{M}_\infty$ then $r = \sum_i a_i \sigma_i(r)$: Since $r \in \mathbf{M}_\infty$, we have $(r, 1) = 0$. On the other hand, since $a_i \in \mathbf{M}_\infty$,

$$\left(\sum_i a_i \sigma_i(r), 1\right) = \sum_i (a_i \sigma_i(r), 1) = \sum_i (a_i, 1)(\sigma_i(r), 1) = 0.$$

Next, for any $w\in W$ we have

$$\begin{aligned}(\sum_i a_i\sigma_i(\tau), a_j w) &= (\sigma_j(\sum_i a_i\sigma_i(\tau)), w)\\ &= (\sum_i \sigma_j(a_i\sigma_i(\tau)), w)\\ &= \sum_i (\sigma_j(a_i\sigma_i(\tau)), w)\\ &= \sum_i [(\sigma_j(a_i)\sigma_i(\tau), w) + (\pi(a_i)\sigma_j(\sigma_i(\tau)), w)]\\ &= \sum_i (\delta_{ji}\sigma_i(\tau), w)\\ &= (\sigma_j(\tau), w)\\ &= (\tau, a_j w).\end{aligned}$$

Since any $u\in W$ is either 1 or of the form $a_j w$ we have $\tau=\sum_i a_i\sigma_i(\tau)$.

(b) $\sum_i a_i t_i=0 \implies t_j=0$ $(1\leqq j\leqq n)$:

$$0=\sigma_j(\sum_i a_i t_i)=\sum_i \sigma_j(a_i t_1)=\sum_i \sigma_j(a_i)t_i+\pi(a_i)\sigma_j(t_i)=\sum_i \delta_{ji}t_i=t_j. \quad \square$$

By this theorem we have the right $\mathbf{S}_\infty$-module isomorphism:

$$\sigma:\mathbf{S}_\infty\oplus \cdots \oplus\mathbf{S}_\infty\to\mathbf{M}_\infty \tag{1.2}$$

such that

$$\sigma(t_1, \ldots, t_n)=a_1 t_1+\cdots+a_n t_n.$$

We have

$$\sigma^{-1}(\tau)=<\sigma_1(\tau), \ldots, \sigma_n(\tau)>.$$

In view of (1.1), the map

$$\tau:\mathbf{S}_\infty\times\cdots\times\mathbf{S}_\infty\to\mathbf{A}_\infty$$

defined by

$$\tau(t_1, \ldots, t_n)=\sigma(t_1, \ldots, t_n)+1$$

is a bijection. Combining (1.1) and (1.2), we have the following set theoretic isomorphism:

$$\mathbf{S}_\infty\simeq 2\times\mathbf{S}_\infty\times\cdots\times\mathbf{S}_\infty \tag{1.3}$$

where

$$\tau\leftrightarrow <\pi(\tau),\sigma_1(\tau), \ldots, \sigma_n(\tau)>.$$

By (1.3), we have the following proposition which is useful for the comparison of two sexps.

Proposition 1.3. *For any* $s, t \in \mathbf{S}_\infty$:

$$s=t \iff \pi(s)=\pi(t),\ \sigma_i(s)=\sigma_i(t) \quad (1 \leqq i \leqq n).$$

(1.3) may be rewritten as:

$$\mathbf{S}_\infty \simeq \mathbf{S}_\infty^n + \mathbf{S}_\infty^n \tag{1.4}$$

where + denotes the (direct) sum of two sets. This isomorphism tells us the basic properties of the data structure $\mathbf{S}_\infty$. Namely, any sexp is an infinite n-ary leaf-free tree which carries one bit information at each node. The *recognizer* π distinguishes atoms from molecules. The *constructor* σ (τ) constructs from given n sexps t_i $(1 \leqq i \leqq n)$ a molecule (atom, resp.) s whose i-th subtree t_i is recovered by the *selector* σ_i.

A sexp s is *invertible* if there exists a t such that $st=ts=1$. Since the t above is unique for an invertible s, it is called the *inverse* of s and is denoted by s^{-1}. We wish to characterize invertible sexps. We need the following lemma.

Lemma 1.4. *If* $r \in \mathbf{M}_\infty$ *then* $r^k \in M_k$ $(k \geqq 0)$ *where*

$$M_k = \{ r \in \mathbf{S}_\infty \mid |w| < k \Longrightarrow (r, w) = 0 \}.$$

Proof. If $k=0$ then $r^0 = 1 \in M_0$. Assume $r^k \in M_k$. Then for any w such that $|w| < k+1$,

$$\begin{aligned}(r^{k+1}, w) &= \sum_{uv=w} (r^k, u)(r, v) \\ &= (r^k, w)(r, 1) + \sum_{\substack{uv=w \\ u \neq w}} (r^k, u)(r, v).\end{aligned}$$

Since $r \in \mathbf{M}_\infty$, we have $(r, 1) = 0$; and if $u \neq w$ we have $(r^k, u) = 0$ by the assumption. Hence $(r^{k+1}, w) = 0$. □

Theorem 1.5. *A sexp is invertible iff it is an atom.*

Proof. ($\Longrightarrow$) If s is invertible, then $ss^{-1}=1$. Hence, $1 = \pi(1) = \pi(ss^{-1}) = \pi(s)\pi(s^{-1})$. Then we have $\pi(s)=1$, so s is an atom.

($\Longleftarrow$) Let s be an atom. We define a molecule r by putting $r = 1 + s$. Then we define a sexp t by:

$$(t, w) = (1 + r + \cdots + r^{|w|}, w).$$

By Lemma 1.4, for any $k \geqq 0$, we have

$$(1 + r + \cdots + r^{|w|+k}, w) = (1 + r + \cdots + r^{|w|}).$$

We have $st = 1$ because:

$$
\begin{aligned}
(st, w) &= \sum_{uv=w} (s, u)(t, v) \\
&= \sum_{uv=w} (1+r, u)(1+r+ \cdots + r^{|v|}, v) \\
&= \sum_{uv=w} (1+r, u)(1+r+ \cdots + r^{|w|}, v) \\
&= (1+r^{|w|+1}, w) \\
&= (1, w) + (r^{|w|+1}, w) \\
&= (1, w).
\end{aligned}
$$

That $ts=1$ holds can be proved similarly. □

2. $\mathbf{S}^{rat}$

We define $\mathbf{S}^{rat}$ as the least subset of $\mathbf{S}_\infty$ such that

(i) $2 \cup \Sigma \subseteq \mathbf{S}^{rat}$,

(ii) $s, t \in \mathbf{S}^{rat} \Longrightarrow s+t \in \mathbf{S}^{rat}$,

(iii) $s, t \in \mathbf{S}^{rat} \Longrightarrow st \in \mathbf{S}^{rat}$,

(iv) $s \in \mathbf{S}^{rat} \cap \mathbf{M}_\infty \Longrightarrow (1+s)^{-1} \in \mathbf{S}^{rat}$.

$\mathbf{S}^{rat}$ is a subring of $\mathbf{S}_\infty$. In this section, we will study the relationship between $\mathbf{S}^{rat}$ and finite automata. Here we define an *automaton* (over Σ) as a triple

$$X = < X; \delta_X, \epsilon_X >$$

where

(1) X is a (possibly infinite) nonempty set of *states*,

(2) $\delta_X: X \times W \to X$ is an action of W on X,

(3) $\epsilon_X: X \to 2$.

Let X be an automaton. For each i $(1 \leqq i \leqq n)$, we define the map

$$\sigma_i^X: X \to X$$

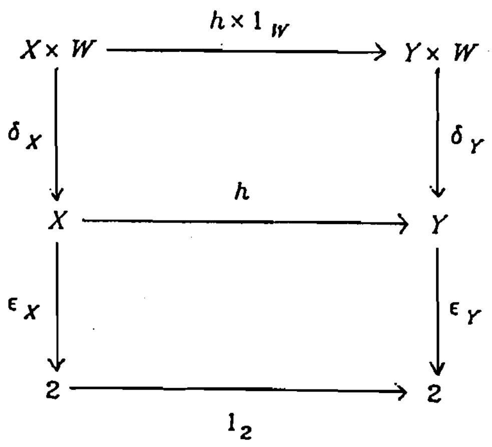
by putting $\sigma_i^X(x) = \delta_X(x, a_i)$. This function determines the transition of states for the input symbol a_i. A state $x \in X$ is considered to be *accepted* if $\epsilon_X(x) = 1$. We now define a function

$$L_X: X \to \mathbf{S}_\infty$$

by putting $(L_X(x), w) = \epsilon_X(\delta_X(x, w))$. $L_X(x)$ may be considered as the language which X, with the initial state x, accepts. Here we also note that $\mathbf{S}_\infty = < \mathbf{S}_\infty; \delta, \pi >$ is an automaton. Moreover, L_X becomes a morphism in the category **Aut** of automata which we now define.

The category **Aut**, by definition, has all automata as its objects. Its morphisms are defined by:

$h \in \mathrm{Hom}(X,Y) \iff h$ is a map for which the diagram below commutes:



Proposition 2.1. $L_X : X \to S_\infty \in \mathrm{Hom}(X, S_\infty)$.

Proof.

$$(\delta(L_X(x), w), u) = (L_X(x), wu) = \epsilon_X(\delta_X(x, wu)),$$

$$(L_X(\delta_X(x,w)), u) = \epsilon_X(\delta_X(\delta_X(x,w), u)) = \epsilon_X(\delta_X(x, wu)).$$

$$\pi(L_X(x)) = (L_X(x), 1) = \epsilon_X(\delta_X(x,1)) = \epsilon_X(x). \quad \Box$$

Proposition 2.2. $L_{S_\infty} : S_\infty \to S_\infty$ *is identity.*

Proof.

$$(L_{S_\infty}(r), w) = \pi(\delta(r, w)) = (\delta(r,w), 1) = (r, w1) = (r, w). \quad \Box$$

Proposition 2.3. $h \in \mathrm{Hom}(X,Y) \Longrightarrow L_Y \circ h = L_X$.

Proof.

$$(L_Y(h(x)), w) = \epsilon_Y(\delta_Y(h(x), w)) = \epsilon_Y(h(\delta_X(x,w))) = \epsilon_X(\delta_X(x,w)) = (L_X(x), w).$$

$\Box$

These propositions yield the following theorem.

Theorem 2.4. S_∞ *is the terminal object of* **Aut**.

Proof. Let X be any automaton. We have $L_X \in \mathrm{Hom}(X, S_\infty)$ by Proposition 2.1. Next, take any $h \in \mathrm{Hom}(X, S_\infty)$. By Proposition 2.2 and Proposition 2.3 for $Y = S_\infty$, we have $L_X = L_{S_\infty} \circ h = 1 \circ h = h$. Thus we have proved that $\mathrm{Hom}(X, S_\infty)$ is a singleton set for any X, i.e., S_∞ is terminal in **Aut**. $\Box$

We now wish to characterize S^{rat} categorically. Let k be an arbitrary natural number. For a ring R, we let $M_k(R)$ denote the matrix ring consisting of $k \times k$ R-matrices. We define a ring homomorphism

$$\Pi_k : M_k(\mathrm{S}_\infty) \to M_k(2)$$

by putting $\Pi_k(S) = (\pi(s_{ij}))$ for $S = (s_{ij}) \in M_k(\mathrm{S}_\infty)$. The set

$$G_k = \Pi_k^{-1}(I_k),$$

where I_k is the $k \times k$ unit matrix, then becomes a monoid under matrix multiplication. Moreover, we have:

Theorem 2.5. *G_k forms a group under matrix multiplication.*

Proof. Let E_{ij} $(1 \leqq i, j \leqq k)$ be the $k \times k$ matrix such that its (i, j) element is 1 and all other elements are 0. For any molecule $r \in \mathbf{M}_\infty$, we put

$$Q_k(i, j; r) = I_k + rE_{ij}.$$

It is easy to see that $Q_k(i, j; r) \in G_k$ and

$$Q_k(i, j; r)^{-1} = \begin{cases} Q_k(i, j; 1 + (1 + r)^{-1}) & \text{if } i = j, \\ Q_k(i, j; r) & \text{if } i \neq j. \end{cases}$$

We can then prove, using usual sweep out method, that the group generated by the set $\{Q_k(i, j; r) \mid 1 \leqq i, j \leqq k, r \in \mathbf{M}_\infty\}$ coincides with G_k. □

Remark. The proof also shows that if $S \in G_k \cap M_k(\mathrm{S}^{rat})$, S^{-1} is also a member of $M_k(\mathrm{S}^{rat})$.

Let $X = \langle X; \delta_X, \epsilon_X \rangle$ be a finite automaton with k states so that $X = \{x_1, \ldots, x_k\}$. For each l $(1 \leqq l \leqq n)$ we define $\tilde{\sigma}_l : \{1, \ldots, k\} \to \{1, \ldots, k\}$ by the condition:

$$\tilde{\sigma}_l(i) = j \iff \sigma_l^X(x_i) = x_j.$$

We then define a $k \times k$ S_∞-matrix $S = (s_{ij})$ by putting

$$s_{ij} = \delta_{ij} + \sum_l \delta_{\tilde{\sigma}_l(i) j} a_l.$$

We note that $S \in G_k \cap M_k(\mathrm{S}^{rat})$. Let X_i $(1 \leqq i \leqq k)$ be k distinct indeterminates and let $\mathbf{x} = {}^t(X_1 \cdots X_k)$. We also put $\mathbf{e} = {}^t(\epsilon_X(x_1) \cdots \epsilon_X(x_k))$. We call the equation:

$$S\mathbf{x} = \mathbf{e} \tag{2.1}$$

the *characteristic equation* of the finite automaton X. By Theorem 2.5 it has the unique solution $\mathbf{x} = S^{-1}\mathbf{e}$. Remark that (2.1) is equivalent to the following system of equations:

$$X_i = a_1 X_{\tilde{\sigma}_1(i)} + \cdots + a_n X_{\tilde{\sigma}_n(i)} + \epsilon_X(x_i) \quad (i = 1, \ldots, k).$$

Theorem 2.6.

$$L_X(x_i) = a_1 L_X(x_{\tilde{\sigma}_1(i)}) + \cdots + a_n L_X(x_{\tilde{\sigma}_n(i)}) + \epsilon_X(x_i) \quad (i = 1, \ldots, k).$$

Proof. Since $L_X \in \mathrm{Hom}(X, \mathrm{S}_\infty)$ we have

$$\sigma_l(L_X(x_i)) = L_X(\sigma_l^X(x_i)) = L_X(x_{\sigma_l(i)}).$$

$$\pi(L_X(x_i)) = \epsilon_X(x_i).$$

Therefore we have:

$$\begin{aligned}\pi(\mathrm{RHS}) &= \pi(a_1 L_X(x_{\sigma_1(i)})) + \cdots + \pi(a_k L_X(x_{\sigma_k(i)})) + \pi(\epsilon_X(x_i)) \\ &= \epsilon_X(x_i) \\ &= \pi(\mathrm{LHS}),\end{aligned}$$

$$\begin{aligned}\sigma_l(\mathrm{RHS}) &= \sigma_l(a_1 L_X(x_{\sigma_1(i)})) + \cdots + \sigma_l(a_k L_X(x_{\sigma_k(i)})) + \sigma_l(\epsilon_X(x_i)) \\ &= L_X(x_{\sigma_l(i)}) \\ &= \sigma_l(L_X(x_i)) \\ &= \sigma_l(\mathrm{LHS}).\end{aligned}$$

This proves LHS=RHS. □

This theorem says that $L_X(x_i)$'s give the solution to the equation (2.1) and hence they are in $\mathbf{S}^{rat}$.

We next show that, conversely, any language in $\mathbf{S}^{rat}$ can be represented by a finite automaton. First we remark that, for a finite set X, 2^X becomes a vector space over $\mathbf{2}$ under the addition defined by:

$$U + V = (U - V) \cup (V - U).$$

If we identify any $x \in X$ with the singleton set $\{x\}$ then X becomes a basis of the vector space 2^X. Let $\mathbf{V}$ be any vector space over $\mathbf{2}$. Then any map $f: X \to \mathbf{V}$ can be uniquely extended to a linear map from 2^X to $\mathbf{V}$. We will denote this extended map also by f.

Let X be any automaton and let Y be a subset of X which is closed under σ_l^X for each l $(1 \leqq l \leqq n)$. Then we can naturally introduce into Y an automaton structure, by restricting that of X to Y, which makes Y a *subautomaton* of X.

We will write

$$X \ni x \models r$$

if x is a state of a finite automaton X and $r = L_X(x)$; and in this case we say that $x \in X$ *realizes* r. Such r's are called *realizable*.

Theorem 2.7. *A sexp r is realizable iff $r \in \mathbf{S}^{rat}$.*

Proof. Only if part follows from the remark below Theorem 2.5.

We prove *if part* by induction on the construction of r.

(i) Since the set $\mathbf{2} \cup \Sigma \subseteq \mathbf{S}_\infty$ is closed under the functions σ_l $(1 \leqq l \leqq n)$, it becomes a finite subautomaton of $\mathbf{S}_\infty$ and each state realizes itself. (cf. Proposition 2.2.)

(ii) $r=s+t$: Assume that $X \ni x_0 \models s$ and $Y \ni y_0 \models t$. We define an automaton Z by putting:

$$Z = X \times Y = \{x \times y \mid x \in X, y \in Y\},$$

$$\delta_Z(x \times y, w) = \delta_X(x, w) \times \delta_Y(y, w),$$

$$\epsilon_Z(x \times y) = \epsilon_X(x) + \epsilon_Y(y).$$

Then by a simple computation we have $L_Z(x \times y) = L_X(x) + L_Y(y)$, so that $Z \ni x_0 \times y_0 \models s+t$.

(iii) $r=st$: Assume that $X \ni x_0 \models s$ and $Y \ni y_0 \models t$. We define an automaton Z by putting:

$$Z = 2^Y \times X = \{\mathrm{y} \times x \mid \mathrm{y} \in 2^Y, x \in X\},$$

$$\sigma_l^Z(\mathrm{y} \times x) = (\sigma_l^Y(\mathrm{y}) + \epsilon_X(x)\sigma_l^Y(y_0)) \times \sigma_l^X(x) \quad (1 \leqq l \leqq n),$$

$$\epsilon_Z(\mathrm{y} \times x) = \epsilon_Y(\mathrm{y}) + \epsilon_X(x)\epsilon_Y(y_0).$$

We show that

$$\tilde{L}(\mathrm{y} \times x) = L_Y(\mathrm{y}) + L_X(x)L_Y(y_0) \quad (\mathrm{y} \times x \in Z)$$

solves the characteristic equation of the automaton Z. I.e., we show that

$$\tilde{L}(z) = a_1 \tilde{L}(\sigma_1^Z(z)) + \cdots + a_n \tilde{L}(\sigma_n^Z(z)) + \epsilon_Z(z) \quad (z \in Z) \tag{2.2}$$

Letting $z = \mathrm{y} \times x$, we compare the LHS and RHS of (2.2) as follows.

$$\begin{aligned}\pi(\mathrm{LHS}) &= \pi(L_Y(\mathrm{y})) + \pi(L_X(x))\pi(L_Y(y_0)) \\ &= \epsilon_Y(\mathrm{y}) + \epsilon_X(x)\epsilon_Y(y_0) \\ &= \epsilon_Z(z) \\ &= \pi(\mathrm{RHS}).\end{aligned}$$

$$\begin{aligned}\sigma_l(\mathrm{LHS}) &= \sigma_l(L_Y(\mathrm{y})) + \sigma_l(L_X(x)L_Y(y_0)) \\ &= L(\sigma_l^Y(\mathrm{y})) + \pi(L_X(x))\sigma_l(L_X(y_0)) + \sigma_l(L_X(x))L_Y(y_0) \\ &= L(\sigma_l^Y(\mathrm{y})) + \epsilon_X(x)L_Y(\sigma_l^Y(y_0)) + L_X(\sigma_l^X(x))L_Y(y_0) \\ &= \tilde{L}(\sigma_l^Z(z)) \\ &= \sigma_l^Z(\mathrm{RHS}) \quad (1 \leqq l \leqq n).\end{aligned}$$

This proves (2.2), so that we have $L_Z(z) = \tilde{L}(z)$. Hence, we have $L_Z(\phi \times x_0) = L_X(x_0)L_Y(y_0) = st$. I.e., $Z \ni \phi \times x_0 \models st$.

(iv) $r=(1+s)^{-1}$, $s \in \mathbf{S}^{rat} \cap \mathbf{M}_\infty$: Assume that $X \ni x_0 \models s$. Since $1 = r^{-1}r = (1+s)r = r+sr$, we have $r = 1+sr$. So, $\sigma_l(r) = \pi(s)\sigma_l(r) + \sigma_l(s)r = \sigma_l(s)r$ $(1 \leqq l \leqq n)$. We define an automaton Z by putting:

$Z = 2^X = \{\mathbf{x} \mid \mathbf{x} \subseteq X\}$,

$\sigma_l^Z(\mathbf{x}) = \sigma_l^X(\mathbf{x}) + \epsilon_X(\mathbf{x})\sigma_l^X(x_0) \quad (1 \leqq l \leqq n)$,

$\epsilon_Z(\mathbf{x}) = \epsilon_X(\mathbf{x})$.

We show that

$$\tilde{L}(\mathbf{x}) = L_X(\mathbf{x})r \quad (\mathbf{x} \in Z)$$

solves the characteristic equation of the automaton Z. I.e., we show the equation:

$$\tilde{L}(\mathbf{x}) = a_1\tilde{L}(\sigma_1^Z(\mathbf{x})) + \cdots + a_n\tilde{L}(\sigma_n^Z(\mathbf{x})) + \epsilon_Z(\mathbf{x}) \quad (\mathbf{x} \in Z) \tag{2.3}$$

We compare the LHS and RHS of (2.3) as follows.

$$\pi(\mathrm{LHS}) = \pi(L_X(\mathbf{x}))\pi(r) = \epsilon_X(\mathbf{x}) = \epsilon_Z(\mathbf{x}) = \pi(\mathrm{RHS}),$$

$$\begin{aligned}
\sigma_l(\mathrm{LHS}) &= \sigma_l(L_l^X(\mathbf{x})r) \\
&= \sigma_l(L_X(\mathbf{x}))r + \pi(L_X(\mathbf{x}))\sigma_l(r) \\
&= L_X(\sigma_l^X(\mathbf{x}))r + \epsilon_X(\mathbf{x})\sigma_l(s)r \\
&= (L_X(\sigma_l^X(\mathbf{x})) + \epsilon_X(\mathbf{x})L_X(\sigma_l^X(x_0)))r \\
&= L_X(\sigma_l^X(\mathbf{x}) + \epsilon_X(\mathbf{x})\sigma_l^X(x_0))r \\
&= \tilde{L}(\sigma_l^X(\mathbf{x}) + \epsilon_X(\mathbf{x})\sigma_l^X(x_0)) \\
&= \tilde{L}(\sigma_l^Z(\mathbf{x})) \\
&= \sigma_l(\mathrm{RHS}).
\end{aligned}$$

This proves (2.3), so that we have $L_Z(\mathbf{x}) = \tilde{L}(\mathbf{x})$. Hence we have $Z \ni x_0 \models L_Z(x_0) = \tilde{L}(x_0) = L_X(x_0)r = sr = 1 + r$. By (ii) above, r is also realizable. □

Corollary 2.8. $r \in \mathbf{S}^{ral} \Longrightarrow \sigma_l(r) \in \mathbf{S}^{ral} \quad (1 \leqq l \leqq n)$.

Proof. By Theorem 2.7, we can find X and x such that $X \ni x \models r$. Then we have $X \ni \sigma_l^X(x) \models \sigma_l(r)$. Hence $\sigma_l(r) \in \mathbf{S}^{ral}$. □

Remark. It is possible to prove Corollary 2.8 directly by induction on the construction of r.

Since $\mathbf{S}^{ral}$ is closed under σ_l for each l $(1 \leqq l \leqq n)$, $\mathbf{S}^{ral}$ becomes a subautomanton of $\mathbf{S}_\infty$. Although $\mathbf{S}^{ral}$ is not a finite automaton, it is a *locally* finite automaton in the sense of the following definition.

Definition. An automaton $X = \langle X; \delta, \epsilon \rangle$ is *locally finite* iff the set $X|x = \{y \mid y = \delta(x, w) \text{ for some } w \in W\}$ is finite for all $x \in X$.

We will denote by $\mathbf{Aut}^{ral}$ the full subcategory of $\mathbf{Aut}$ consisting of all the locally finite automata. We have the following theorem.

Theorem 2.9. *$\mathbf{S}^{ral}$ is the terminal object of* $\mathbf{Aut}^{ral}$.

Proof. We first prove the claim that S^{rat} is a locally finite automaton. Suppose that $r \in S^{rat}$. By Theorem 2.7, we can find X and x such that $X \ni x \models r$. Since Im L_X is finite and closed under σ_i $(1 \leqq i \leqq n)$, the set $X|x$ is also finite. This proves the claim.

Next, let X be an arbitrary locally finite automaton, and consider the map $L_X: X \to S_\infty$. For any $x \in X$, $X|x$ becomes a finite subautomaton of X. Then we have $L_X(x) = L_{X|x}(x) \in S^{rat}$, so that we may regard L_X as the map $L_X: X \to S^{rat}$. Now the theorem can be proved similarly as Theorem 2.4. □

3. S

We define S as the least subset of S_∞ such that

(i) $2 \cup \Sigma \subseteq S$,

(ii) $s, t \in S \Longrightarrow s + t \in S$,

(iii) $s, t \in S \Longrightarrow st \in S$.

According to this definition of S, S becomes a subring of S_∞. Elements of S are called *finite* sexps. We can establish the set theoretic isomorphism:

$$S \simeq S^n + S^n \tag{3.1}$$

similarly as (1.4). Just as (1.4) expressed some characteristics of S_∞, this equation says that S is a data structure equipped with the recognizer π, constructors σ, τ and selectors σ_i $(1 \leqq i \leqq n)$. Furthermore, it is easy to verity that S can be characterized as the least subset of S_∞ such that

(1) $0 \in S$,

(2) $t_1, \ldots, t_n \in S \Longrightarrow \sigma(t_1, \ldots, t_n) \in S$,

(3) $t_1, \ldots, t_n \in S \Longrightarrow \tau(t_1, \ldots, t_n) \in S$.

Namely, any finite sexp can be constructed from the initial sexp 0 by applying σ and τ finitely many times. This characterization of S gives the following Proposition, which explains our naming of finite sexps.

Proposition 3.1. *A sexp s is finite iff* supp(*s*) *is a finite set.*

We remark that Scott[4, p. 96] also discusses the domain equation of the form (3.1), and gives a solution for it as a *neighbourhood system.* In Scott[4], the interpretations of sums and products are slightly different from ours, so that *total* elements in his solution corresponds to symbolic expressions in our sense. He also points out that any *eventually periodic* total tree (which corresponds to our rational sexp) represents an automaton such that each state of the automaton realizes itself.

Finally, we remark that in case $\Sigma = \{a_1, a_2\}$ finite sexps are precisely the symbolic expressions in the sense of Sato[2] and Sato and Hagiya[3]. In [2] and [3], the functions σ, τ, σ_1 and σ_2 are respectively called *cons, snoc, car* and *cdr* following the tradition of Lisp.

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