Bounded Modal Logic

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Abstract

Under the Curry–Howard isomorphism, the syntactic structure of programs can be modeled using birelational Kripke structures equipped with intuitionistic and modal relations. Intuitionistic relations capture scoping through persistence, reflecting the availability of resources from outer scopes, while modal relations model resource isolation introduced for various purposes.

Traditional modal languages, however, describe only modal transitions and thus provide limited support for expressing fine-grained control over resource availability. Motivated by this limitation, we introduce *Bounded Modal Logic (**BML**)*, an experimental extension of constructive modal logic whose language explicitly accounts for both intuitionistic and modal transitions.

We present a natural-deduction proof system and a Kripke semantics for **BML**, together with a Curry–Howard interpretation via a corresponding typed lambda-calculus. We establish metatheoretic properties of the calculus, showing that **BML** forms a well-disciplined logical system. This provides theoretical support for our proposed perspective on fine-grained resource control in programming languages.

2012 ACM Subject Classification Theory of computation → Modal and temporal logics

Keywords and phrases modal logic, modal lambda-calculus, Kripke semantics

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

Funding *Yuito Murase*: Supported by JSPS KAKENHI Grant Number JP24KJ1363.

1 Introduction

Kripke Structure in Lambda-Terms

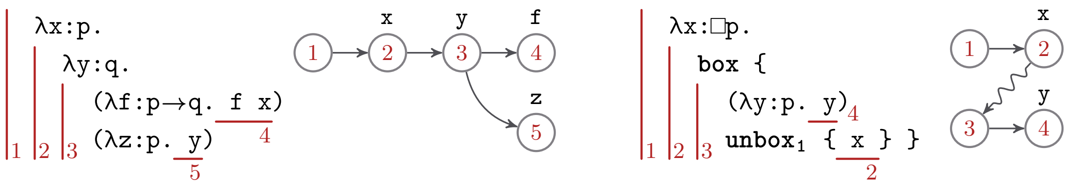
The Curry–Howard isomorphism points out the correspondence between logic and programs [9, 31]. While this correspondence is observed via those between proof systems and typed lambda-calculi, we can apply semantical notions from logic to typed lambda-calculi. For example, Mitchell and Moggi [19] generalized Kripke semantics for intuitionistic logic to understand typed lambda-terms in simply typed lambda-calculus.

More informally, the scoping structure of programs can be understood through the lens of Kripke semantics for intuitionistic logic. In this view, scope inclusion induces an intuitionistic accessibility relation satisfying persistency: variables defined at an outer scope remain available at inner scopes. Figure 1a illustrates this correspondence.

In advanced settings, various modal lambda-calculi have been developed to model computational effects [20], information-flow control [1, 29], multi-stage programming [6, 7], distributed programming [23] and functional reactive programming [3, 14] among others. They use modal types as a means to introduce appropriate isolation for resources.

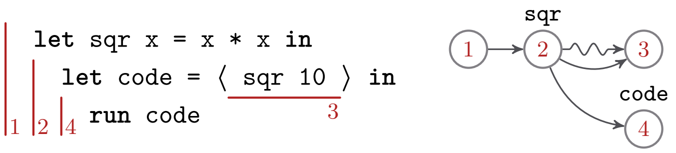
We can also observe Kripke structures in modal lambda-calculi. As an example, we consider Kripke/Fitch-style **S4** modal lambda-calculi [5, 7, 38], whose correspondence with Kripke semantics for constructive modal logic has been studied in the literature. Figure 1b illustrates a modal lambda-term and its associated structure. In addition to intuitionistic transitions corresponding to scope inclusion, the structure includes modal transitions induced

42nd Conference on Very Important Topics (CVIT 2016).
Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:1–23:40
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



(a) Simply typed lambda-calculus.

(b) Kripke-style modal lambda-calculus [7].



(c) MetaML-style staged program.

Figure 1 Kripke structure of lambda-terms. Here, $\longrightarrow$ and $\rightsquigarrow$ represent intuitionistic and modal relations, respectively.

by box and unbox constructs. These modal transitions are non-persistent and introduce isolation between scopes, capturing controlled communication of resources in programs.

From this viewpoint, we can regard birelational Kripke structure as capturing resource control taking place in lambda-terms: intuitionistic relations coincide with nested scopes allowing persistent use of resources, while modal transitions coincide with resource isolation where communications need to be explicitly mentioned by modal operators.

Limitation of the Traditional Modal Language

The traditional modal language refers only to modal relations, and this is not sufficient to describe richer resource control for some practical programs. We consider Multi-Stage Programming (MSP), whose correspondence to modal logic is well studied [6, 7, 24, 37, 39]. In particular, we are interested in modal typing disciplines for MetaML-style MSP [21, 33–35].

The key charasteristics of MetaML-style MSP are as follows: **1.** code fragments are represented by quasi-quotation syntax with brackets $\langle \cdots \rangle$ and splices ~$\cdots$; **2.** quotations allow using variables defined outside; and **3.** code fragments can be evaluated at runtime via `run`. Figure 1c is such an example of MetaML-style stage program. We define a variable `code` as an open code fragment with a variable `sqr`, and evaluate it under the scope of `sqr` by `run`. We can also consider a birelational Kripke structure for this program, as presented on the right side. Its structure is more complex than that of previous examples. We see both intuitionistic and modal relations from 2 to 3: the intuitionistic relation is required to use `sqr` via persistency, while the modal one is introduced by the quotation. Here, we can regard modal transitions as isolation between stages.

The question here is what the type for `⟨ sqr 10 ⟩` would be in this example. **S4** modality seems appropriate because we have `run` [7]. However, `□int` does not type `⟨ sqr 10 ⟩`: this type asserts that it can pass an integer value via *any* modal transitions, but this code fragment is only valid under the scope of `sqr`. Hence, we want to relax modal types to state "we can embed an integer expression to a scope where `sqr` is available via persistency."

It is worth noting that such patterns are observed not only in multi-stage programming. In the context of distributed programming, there can be values that are sent to agents satisfying specific resource constraints. In the context of computational effects, we might have effects that are valid under specific scopes. Here, we want to develop an extension to the traditional

$$\forall\gamma :\succeq \mathbf{!}.\ \Box^{\succeq\gamma}(A\to B)\to\Box^{\succeq\gamma}A\to\Box^{\succeq\gamma}B \tag{K}$$

$$\Box^{\succeq\mathbf{!}}A\to A \tag{T}$$

$$\forall\gamma :\succeq \mathbf{!}.\ \Box^{\succeq\gamma}\Box^{\succeq\gamma}A\to\Box^{\succeq\gamma}A \tag{4^{-1}}$$

$$\forall\gamma :\succeq \mathbf{!}.\ \Box^{\succeq\gamma}A\to\Box^{\succeq\gamma}\Box^{\succeq\gamma}A \tag{4}$$

$$\forall\gamma_1 :\succeq \mathbf{!}.\ (\Box^{\succeq\gamma_1}A\to\forall\gamma_2 :\succeq\gamma_1.\ \Box^{\succeq\gamma_2}A) \tag{Mon}$$

$$\forall\gamma_1 :\succeq \mathbf{!}.\ ((\forall\gamma_2 :\succeq\gamma_1.\ \Box^{\succeq\gamma_2}A)\to\Box^{\succeq\gamma_1}A) \qquad \text{where } \gamma_2\notin\mathbf{FC}(A) \tag{Mon^{-1}}$$

$$\forall\gamma_1 :\succeq \mathbf{!}.\ (\forall\gamma_2 :\succeq\gamma_1.\ \Box^{\succeq\gamma_2}A\to\Box^{\succeq\gamma_2}B)\to\Box^{\succeq\gamma_1}(A\to B) \qquad \text{where } \gamma_2\notin\mathbf{FC}(A)\cup\mathbf{FC}(B) \tag{K^{-1*}}$$

Figure 2 Tautologies in **BML**. **!** represents global scope, and **FC**(A) is a set of classifiers occurs freely in A.

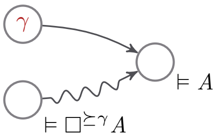
modal language, which is applicable to patterns of this kind.

Modal Logic Reasoning Persistency

As an answer to the problem above, we propose *Bounded Modal Logic (***BML***)*. The language of **BML** is defined as follows:

$$A,\ B ::= p \mid A\to B \mid \Box^{\succeq\gamma}A \mid \forall\gamma_1 :\succeq \gamma_2.\ A$$

We see objects like γ, which we call *classifiers*. Classifiers are atomic objects that represent elements in a Kripke structure, which we can regard variable scopes. We have a special classifier **!** that represents the global scope. Then, *bounded modality* $\Box^{\succeq\gamma}A$ states that it can pass a term of A via modal transition, into scopes where we can use resources from γ via persistency, like the following figure:



Bounded modality allows ones to type expressions with relaxed isolation. In the context of MSP, $\Box^{\succeq\gamma}A$ can be read as a type for "code fragments of A expression which is valid under the scope γ," allowing to typing program in Figure 1c.

On the other hand, *polymorphic classifier quantifier* $\forall\gamma_1 :\succeq \gamma_2.\ A$ introduces a quantification over γ_1 with lower bound γ_2 with regard to intuitionistic transition. Such a quantifier is useful to generalize functions with classifiers: $\forall\gamma_1 :\succeq \gamma_2.\ (\Box^{\succeq\gamma_1}A\to\Box^{\succeq\gamma_1}A)$ can be interpreted as a type for function over code fragments with arbitrary scope γ_1, which is nested within γ_2. Thus, **BML** has an aspect of first-order predicate logic whose domain is a birelational Kripke structure. Note that atomic propositions in **BML** are only propositional variables, and relational statements like $\gamma_1 \preceq \gamma_2$ or $\gamma_1 \sqsubseteq \gamma_2$ do not form a proposition on their own, unlike ordinary predicate logic. As we shall see later, those assertions will be treated as first-class judgments in our proof system.

In this way, we can regard **BML** as combining modal logic and first-order predicate logic in a novel manner. We give a list of valid propositions of **BML** in Figure 2. K, T, 4^{-1}, 4 are generalization of common modal axioms. Mon, Mon^{-1} show that $\Box^{\succeq\gamma}A$ and $\forall\gamma' :\succeq \gamma.\ \Box^{\succeq\gamma'}A$ are equivalent, characterizing persistent nature of classifiers. K^{-1*} is one of interesting examples using polymorphic classifier quantifiers, which acts like an inverse of K.

Our Contribution

This paper aims to establish a theoretical foundation for **BML** as a constructive modal logic, together with its computational counterpart. We first introduce a formal definition of a birelational Kripke structure, called a **BML**-structure, which serves as the underlying semantic and conceptual basis for the logic (Section 2). Building on this structure, we define a natural-deduction proof system (Section 3) and a Kripke semantics (Section 4) for **BML**, and prove their correspondence via soundness and completeness (Section 5). We then introduce a corresponding modal lambda-calculus and study its metatheory, thereby providing a computational interpretation of **BML** (Section 6). Finally, we prove that **BML** generalizes **CS4** from both semantic and proof-theoretic perspectives (Section 7).

Note that we omit detailed definitions and proofs from the paper due to page limitation. One can find such details in Appendix.

2 Syntactic Structure of Modal Lambda-Terms

Before introducing **BML**, we formally define birelational Kripke structure – we call it **BML**-structure – that captures the structure of modal lambda-terms.

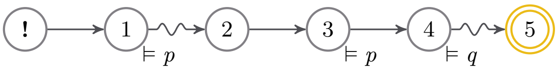
▶ **Definition 2.1** (**BML**-Structure). *A* **BML***-structure is a quintuple* $\langle D, \preceq, \sqsubseteq, V, \mathbf{!}\rangle$ *where*
- $\langle D, \preceq\rangle$ *is a preordered set with*
- *a root* $\mathbf{!}$*, the least element of* D *with respect to* $\preceq$*;*
- $\sqsubseteq$ *is a preorder on* D *with* stability[1] *condition* $(\preceq) \subseteq (\sqsubseteq)$*, or equivalently: left-stability* $(\preceq ; \sqsubseteq) \subseteq (\sqsubseteq)$ *and right-stability* $(\sqsubseteq ; \preceq) \subseteq (\sqsubseteq)$*;*
- V *assigns each atom* p *to an upward-closed subset of* D*.*

We regard **BML**-structure as the essence of syntactic structure of programs, where the intuitionistic relation $\preceq$ denotes scope inclusion and the modal relation $\sqsubseteq$ denotes resource isolation. D is a set of scopes or locations in programs. $\mathbf{!}$ represents the global scope, which behaves as a bottom element with regard to $\preceq$. We impose the stability condition for the modal relation in parity with the proof system that we introduce in Section 3.

Before we define proof system of **BML**, we discuss how we can find **BML**-structure in Kripke/Fitch-style proof systems [5, 7, 38], which inspired the design of our proof system for **BML**. This introduces several key notions that we will use in the next section. We define a proof system for constructive variant of **S4** in Kripke/Fitch-style in the literature. We derive propositions via the judgment $\Gamma \vdash_{\mathbf{S4}} A$, where a context Γ has the following structure.

$$\Gamma, \Delta ::= \varepsilon \mid \Gamma, A \mid \Gamma, \blacktriangleright$$

Davies and Pfenning [7] mention that $\blacktriangleright$ represents modal transition, and propositions between each $\blacktriangleright$ represent assumptions holding at a specific world.[2] We elaborate this idea to make a correspondence between a context and a **BML**-structure. For example, a context $p, \blacktriangleright, p, q, \blacktriangleright$ corresponds to the **BML**-structure below:



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[1] The term ‘stable’ is borrowed from Stell, Schmidt, and Rydeheard [32].

[2] To be precise, Davies and Pfenning [7] introduced a stack structure over contexts to represent modal transitions between contexts. In our definition, we can regard $\blacktriangleright$ to delimit contexts. With regard to Fitch-style proof systems [5, 38], we can identify $\blacktriangleright$ with 🔓 in their definition of contexts.

$\longrightarrow$ and $\rightsquigarrow$ represent $\preceq$ and $\sqsubseteq$, respectively. Here, we can regard a context to carry three kinds of information:

1. A **BML**-structure itself;
2. Assumptions holding at each element of the **BML**-structure; and
3. The current position in the **BML**-structure where deduction is performed (drawn as a yellow double circle).

Particularly, we use the notion of the current position of a **BML**-structure; we simply call it a *position* in the rest of the paper. Then, each item in a context works in the following way:

- An empty context ε corresponds to a **BML**-structure with a single element $\mathbf{!}$. No assumption holds there, and the position is $\mathbf{!}$.
- When adding an assumption A to Γ, it introduces a new element with an assumption A. The position moves to the new element, and it introduces $\preceq$ along with the movement.
- When adding $\blacktriangleright$ to Γ, it introduces a new element, and the position moves to it. It also introduces $\sqsubseteq$ along with the movement.

We can informally understand derivation rules with respect to **BML**-structure of contexts, regarding them as Kripke models for intuitionistic modal logic [2,30]. We explain selected three rules from the viewpoint of **BML**-structure:

$$\textbf{Hyp}\ \frac{\blacktriangleright \notin \Gamma_2}{\Gamma_1,\, A,\, \Gamma_2 \vdash_{\mathbf{S4}} A} \qquad \Box\textbf{-I}\ \frac{\Gamma,\, \blacktriangleright \vdash_{\mathbf{S4}} A}{\Gamma \vdash_{\mathbf{S4}} \Box A} \qquad \Box\textbf{-E}\ \frac{\Gamma_1 \vdash_{\mathbf{S4}} \Box A}{\Gamma_1,\, \Gamma_2 \vdash_{\mathbf{S4}} A}$$

Hyp: As Γ_2 does not include $\blacktriangleright$, the relations introduced by Γ_2 are all $\preceq$. As $\preceq$ is reflexive and transitive, we can leverage persistency to conclude that A holds at the position of $\Gamma_1,\, A,\, \Gamma_2$.

□-I: The premise $\Gamma,\, \blacktriangleright \vdash_{\mathbf{S4}} A$ states that A holds at the element that is reachable by $\sqsubseteq$ from the position of Γ. Hence, we can conclude $\Box A$ at the position of Γ.

□-E: As the relations introduced by Γ_2 include both $\preceq$ and $\sqsubseteq$, we get a single transition with $\sqsubseteq$ using the stability condition and reflexivity/transitivity of $\sqsubseteq$. Hence, we can safely conclude A at the position of $\Gamma_1,\, \Gamma_2$.

When we assign lambda-terms to these rules, we can regard contexts to include the essence of the structure of lambda-terms. Thus, it is natural to find correspondence between contexts and **BML**-structure. In Section 3 and Section 6, we develop a richer context structure and lambda-terms for **BML**.

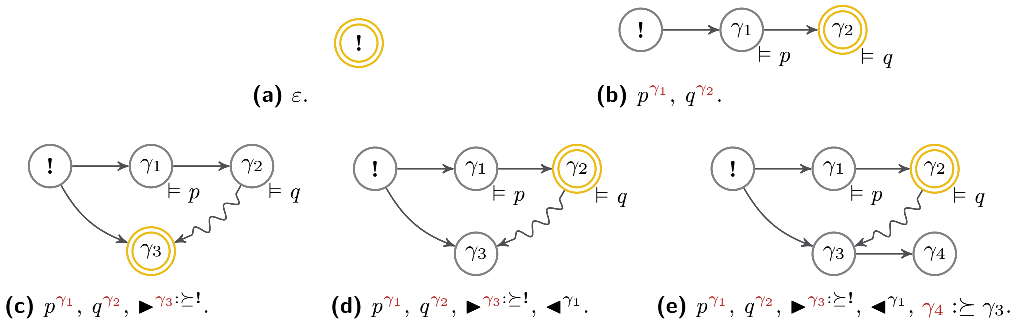
3 Natural Deduction

Having introduced the underlying **BML**-structure as a conceptual basis, we now formalize reasoning over it by presenting a natural-deduction proof system. This proof system is designed to internalize the intuitions observed in Kripke/Fitch-style proof systems. First, we annotate *classifiers* to each item of a context, following the tradition of *labelled proof systems* [8,25,28]. This allows us to refer to elements in a **BML**-structure.

$$\Gamma,\, \Delta ::= \varepsilon \mid \Gamma,\, A^{\gamma} \mid \Gamma,\, \blacktriangleright^{\gamma}$$

We also use an *initial* classifier $\mathbf{!}$ to refer to an empty context. We then extend the structure of contexts as follows (For clarity, we color classifiers if they are newly introduced):

$$\Gamma,\, \Delta ::= \varepsilon \mid \Gamma,\, A^{\color{red}\gamma} \mid \Gamma,\, \blacktriangleright^{{\color{red}\gamma_1} :\succeq \gamma_2} \mid \Gamma,\, \blacktriangleleft^{\gamma} \mid \Gamma,\, {\color{red}\gamma_1} :\succeq \gamma_2$$



(a) ε.　**(b)** p^{γ_1}, q^{γ_2}.

(c) p^{γ_1}, q^{γ_2}, $\blacktriangleright^{\gamma_3 :\succeq !}$.　**(d)** p^{γ_1}, q^{γ_2}, $\blacktriangleright^{\gamma_3 :\succeq !}$, $\blacktriangleleft^{\gamma_1}$.　**(e)** p^{γ_1}, q^{γ_2}, $\blacktriangleright^{\gamma_3 :\succeq !}$, $\blacktriangleleft^{\gamma_1}$, $\gamma_4 :\succeq \gamma_3$.

Figure 3 **BML** contexts and its corresponding **BML**-structure. $\longrightarrow$ and $\rightsquigarrow$ indicate intuitionistic and modal transitions, respectively. Each yellow double circle indicates the current position of each context.

The behavior of an empty context and A^{γ} does not change. $\blacktriangleright^{\gamma_2 :\succeq \gamma_1}$ introduces a intuitionistic relation from γ_1 to γ_2, in addition to a modal relation. $\blacktriangleleft^{\gamma_1}$ is a new item that only moves position to γ_1, which requires that there is a modal transition in the opposite direction of the movement. $\gamma_2 \succeq \gamma_1$ is also a new item that introduces a new element γ_2, and an intuitionistic relation from γ_1 to γ_2. We summarize the behavior of each item in Table 1. Figure 3 provides concrete examples for contexts and corresponding **BML**-structures.

We proceed with formal definitions. First we define $\mathrm{pos}(\Gamma)$, the position of Γ.

▶ **Definition 3.1** (Position of Contexts).

$$\mathrm{pos}(\varepsilon) = \mathbf{!} \qquad \mathrm{pos}(\Gamma,\, x :^{\gamma} A) = \gamma \qquad \mathrm{pos}(\Gamma,\, \blacktriangleright^{\gamma_1 :\succeq \gamma_2}) = \gamma_1$$

$$\mathrm{pos}(\Gamma,\, \blacktriangleleft^{\gamma}) = \gamma \qquad \mathrm{pos}(\Gamma,\, \gamma_1 :\succeq \gamma_2) = \mathrm{pos}(\Gamma)$$

For the sake of space, we introduce a shorthand notation for positions.

▶ **Notation.** *We write* Γ^{γ} *to represent* Γ *with its position* γ*. When we write a context with multiple meta-variables like* $\Gamma_1^{\gamma_1}, \Gamma_2^{\gamma_2}$*, then it means that* $\mathrm{pos}(\Gamma_1) = \gamma_1$ *and* $\mathrm{pos}(\Gamma_1, \Gamma_2) = \gamma_2$ *hold. Note that it does not mean* $\mathrm{pos}(\Gamma_2) = \gamma_2$ *because* Γ_2 *can be empty.*

We write $\mathbf{Dom}_{\mathrm{C}}(\Gamma)$ for a set of classifiers defined in Γ, corresponding to elements in a **BML**-structure. Then, $\Gamma \vdash \gamma_1 \preceq \gamma_2$ and $\Gamma \vdash \gamma_1 \sqsubseteq \gamma_2$ describes the intuitionistic and modal relations between classifiers, whose derivation rules can be found in Figure 4. $\trianglelefteq$-Refl and $\trianglelefteq$-Trans describes reflexive and transitive nature of these relations. $\sqsubseteq$-Lift states that $\preceq$ can

Table 1 Summary of the context syntax: what happens when Γ is extended with each item. Colored classifiers indicate that they are introduced by the items.

Item	Moves position? (to)	Assumptions added			Consumed by
		($\preceq$)	($\sqsubseteq$)	Proposition	
A^{γ}	Yes (γ)	$\mathrm{pos}(\Gamma) \preceq \gamma$	—	A holds at γ	($\to$-I)
$\blacktriangleright^{\gamma_2 :\succeq \gamma_1}$	Yes (γ_2)	$\gamma_1 \preceq \gamma_2$	$\mathrm{pos}(\Gamma) \sqsubseteq \gamma_2$	—	($\Box$-I)
$\blacktriangleleft^{\gamma}$	Yes (γ)	—	—$^{1)}$	—	($\Box$-E)
$\gamma_2 \succeq \gamma_1$	No	$\gamma_1 \preceq \gamma_2$	—	—	($\forall$-I)

1) $\gamma \sqsubseteq \mathrm{pos}(\Gamma)$ is not added as an assumption, but is required to be deduced from Γ.

1. $\trianglelefteq \in \{\preceq, \sqsubseteq\}$. **2.** $\vdash \Gamma : \mathbf{ctx}$ is assumed.

$$\textbf{$\trianglelefteq$-Refl}\quad \frac{\gamma \in \mathbf{Dom}_{\mathrm{C}}(\Gamma)}{\Gamma \vdash \gamma \trianglelefteq \gamma} \qquad \textbf{$\trianglelefteq$-Trans}\quad \frac{\Gamma \vdash \gamma_1 \trianglelefteq \gamma_2 \quad \Gamma \vdash \gamma_2 \trianglelefteq \gamma_3}{\Gamma \vdash \gamma_1 \trianglelefteq \gamma_3} \qquad \textbf{$\sqsubseteq$-Lift}\quad \frac{\Gamma \vdash \gamma_1 \preceq \gamma_2}{\Gamma \vdash \gamma_1 \sqsubseteq \gamma_2}$$

$$\textbf{$\preceq$-Hyp}\quad \frac{}{\Gamma_1^{\gamma_1}, A^{\gamma_2}, \Gamma_2 \vdash \gamma_1 \preceq \gamma_2} \qquad \textbf{$\preceq$-Cls}\quad \frac{\gamma_1 :\succeq \gamma_2 \in \Gamma}{\Gamma \vdash \gamma_2 \preceq \gamma_1} \qquad \textbf{$\sqsubseteq$-$\blacktriangleright$}\quad \frac{}{\Gamma_1^{\gamma_1}, \blacktriangleright^{\gamma_2 :\succeq \gamma_3}, \Gamma_2 \vdash \gamma_1 \sqsubseteq \gamma_2} \qquad \textbf{$\preceq$-$\blacktriangleright$}\quad \frac{\blacktriangleright^{\gamma_1 :\succeq \gamma_2} \in \Gamma}{\Gamma \vdash \gamma_2 \preceq \gamma_1}$$

Figure 4 Derivation rules for $\Gamma \vdash \gamma_1 \preceq \gamma_2$ and $\Gamma \vdash \gamma_1 \sqsubseteq \gamma_2$.

$$\textbf{WF-$\blacktriangleleft$}\quad \frac{\vdash \Gamma^{\gamma} : \mathbf{ctx} \quad \Gamma^{\gamma} \vdash \delta \sqsubseteq \gamma}{\vdash \Gamma^{\gamma}, \blacktriangleleft^{\delta} : \mathbf{ctx}} \qquad \textbf{Hyp}\quad \frac{A^{\gamma_1} \in \Gamma^{\gamma_2} \quad \Gamma^{\gamma_2} \vdash \gamma_1 \preceq \gamma_2}{\Gamma^{\gamma_2} \vdash A} \qquad \textbf{$\to$-I}\quad \frac{\Gamma, A_1{}^{\gamma} \vdash A_2 \quad \gamma_1 \notin \mathbf{FC}(A_2)}{\Gamma \vdash A_1 \to A_2}$$

$$\textbf{$\to$-E}\quad \frac{\Gamma \vdash A_1 \to A_2 \quad \Gamma \vdash A_1}{\Gamma \vdash A_2} \qquad \textbf{$\Box$-I}\quad \frac{\Gamma, \blacktriangleright^{\gamma_1 :\succeq \gamma_2} \vdash A \quad \gamma_1 \notin \mathbf{FC}(A)}{\Gamma \vdash \Box^{\succeq \gamma_2} A} \qquad \textbf{$\Box$-E}\quad \frac{\Gamma^{\gamma_1}, \blacktriangleleft^{\gamma_2} \vdash \Box^{\succeq \gamma_3} A \quad \Gamma^{\gamma_1} \vdash \gamma_3 \preceq \gamma_1}{\Gamma^{\gamma_1} \vdash A}$$

$$\textbf{$\forall$-I}\quad \frac{\Gamma, \gamma_1 :\succeq \gamma_2 \vdash A}{\Gamma \vdash \forall \gamma_1 :\succeq \gamma_2.\ A} \qquad \textbf{$\forall$-E}\quad \frac{\Gamma \vdash \forall \gamma_1 :\succeq \gamma_2.\ A \quad \Gamma \vdash \gamma_2 \preceq \gamma_3}{\Gamma \vdash A[\gamma_1 := \gamma_3]}$$

Figure 5 Curated rules for $\vdash \Gamma : \mathbf{ctx}$ and $\Gamma \vdash A$.

lift to $\sqsubseteq$, which corresponds to the stability condition in **BML**-structure. The rest of the rules introduce $\preceq$ and $\sqsubseteq$ based on each item of a context as described in Table 1.

$\vdash \Gamma : \mathbf{ctx}$ and $\Gamma \vdash A : \mathbf{prop}$ states well-formedness of Γ and A, respectively. Most of derivation rules for these judgments ensure occurrences of classifiers in Γ and A are well defined, and we omit rules. One exception is WF-$\blacktriangleleft$ in Figure 5, which ensures the context $\Gamma^{\gamma}, \blacktriangleleft^{\delta}$ to satisfy $\Gamma^{\gamma} \vdash \delta \sqsubseteq \gamma$.

The judgment $\Gamma \vdash A$ asserts truth of A under the context Γ. Figure 5 lists derivation rules. As discussed in Kripke/Fitch-style proof systems, we can understand these derivation rules via **BML**-structure. Hyp explicitly requires an intuitionistic transition $\gamma_1 \preceq \gamma_2$ to use A at γ_1 via persistency. Introduction rules $\to$-I, $\Box$-I and $\forall$-I states that implication, bounded modality and polymorphic classifier quantifiers corresponds to the structure of A^{γ}, $\blacktriangleright^{\gamma_1 :\succeq \gamma_2}$ and $\gamma_1 \succeq \gamma_2$, respectively. $\Box$-E uses the structure of $\blacktriangleleft$ to get a modal relation from γ_2 to γ_1. It also requires a condition $\gamma_3 \preceq \gamma_1$, which is required by the bound. As examples for derivations, Figure 6 provides derivations for K^{-1*} and T.

Finally, we confirm that truth of propositions are persistent via intuitionistic transitions.

▶ **Lemma 3.2** (Persistency). *If* $\Gamma_1^{\gamma_1} \vdash A$ *and* $\Gamma_1^{\gamma_1}, \Gamma_2^{\gamma_2} \vdash \gamma_1 \preceq \gamma_2$*, then* $\Gamma_1^{\gamma_1}, \Gamma_2^{\gamma_2} \vdash A$.

4 Kripke Semantics

To define Kripke semantics for **BML**, we apply the idea of that for intuitionistic first-order logic [30], regarding **BML**-structure as its domain. Thus, we define **BML**-modal as a family of **BML**-structures that captures growing domain structure along with $\preccurlyeq$.

▶ **Definition 4.1** (**BML**-Model). *A* **BML**-*model is a triple* $\langle W, \preccurlyeq, \{M_w\}_{w \in W} \rangle$ *where*

$$\Gamma_2 = \gamma_1 :\succeq \mathbf{!},\ (\forall\gamma_2 :\succeq \gamma_1.\ \Box^{\succeq\gamma_2}A \to \Box^{\succeq\gamma_2}B)^{\gamma_3} \qquad \Gamma_3 = \blacktriangleright^{\gamma_4 :\succeq \gamma_1},\ A^{\gamma_5},\ \blacktriangleleft^{\gamma_3}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_2,\ \Gamma_3 \vdash \forall\gamma_2 :\succeq \gamma_1.\ (\Box^{\succeq\gamma_2}A \to \Box^{\succeq\gamma_2}B)}\,\text{Hyp}}{\Gamma_2,\ \Gamma_3 \vdash \Box^{\succeq\gamma_5}A \to \Box^{\succeq\gamma_5}B}\,\forall\text{-E} \quad \dfrac{\dfrac{}{\Gamma_2,\ \Gamma_3,\ \blacktriangleright^{\gamma_6 :\succeq \gamma_5} \vdash A}\,\text{Hyp}}{\Gamma_2,\ \Gamma_3 \vdash \Box^{\succeq\gamma_5}A}\,\Box\text{-I}}{\Gamma_2,\ \Gamma_3 \vdash \Box^{\succeq\gamma_5}B}\,\to\text{-E}}{\Gamma_2,\ \blacktriangleright^{\gamma_4 :\succeq \gamma_1},\ A^{\gamma_5} \vdash B}\,\Box\text{-E}}{\Gamma_2,\ \blacktriangleright^{\gamma_4 :\succeq \gamma_1} \vdash A \to B}\,\to\text{-I}}{\Gamma_2 \vdash \Box^{\succeq\gamma_1}(A \to B)}\,\Box\text{-I}}{\gamma_1 :\succeq \mathbf{!} \vdash (\forall\gamma_2 :\succeq \gamma_1.\ \Box^{\succeq\gamma_2}A \to \Box^{\succeq\gamma_2}B) \to \Box^{\succeq\gamma_1}(A \to B)}\,\to\text{-I}}{\varepsilon \vdash \forall\gamma_1 :\succeq \mathbf{!}.\ (\forall\gamma_2 :\succeq \gamma_1.\ \Box^{\succeq\gamma_2}A \to \Box^{\succeq\gamma_2}B) \to \Box^{\succeq\gamma_1}(A \to B)}\,\forall\text{-I}$$

$$\dfrac{\dfrac{\dfrac{}{(\Box^{\succeq\mathbf{!}}A)^{\gamma_1} \vdash \Box^{\succeq\mathbf{!}}A}\,\text{Hyp}}{(\Box^{\succeq\mathbf{!}}A)^{\gamma_1} \vdash A}\,\Box\text{-E}}{\varepsilon \vdash \Box^{\succeq\mathbf{!}}A \to A}\,\to\text{-I}$$

Figure 6 Derivations for K^{-1*} and T (omitting derivations w.r.t. $\preceq$).

- *$\langle W, \preccurlyeq\rangle$ is a nonempty preordered set;*
- *Each M_w is a* **BML**-*structure $\langle D_w, \preceq_w, \sqsubseteq_w, V_w, \mathbf{!}_w\rangle$; and*
- *If $w_1 \preccurlyeq w_2$, then*
  - *$D_{w_1} \subseteq D_{w_2}$;*
  - *$d_1 \preceq_{w_1} d_2 \implies d_1 \preceq_{w_2} d_2$;*
  - *$d_1 \sqsubseteq_{w_1} d_2 \implies d_1 \sqsubseteq_{w_2} d_2$;*
  - *$V_{w_1}(p) \subseteq V_{w_2}(p)$; and,*
  - *$\mathbf{!}_{w_1} = \mathbf{!}_{w_2}$.*

Given a **BML**-model $\mathfrak{M} = \langle W, \preccurlyeq, \{M_w\}_{w\in W}\rangle$ and $w \in W$, a *w-assignment* ρ is a partial map from the set of all classifiers to D_w with $\mathbf{!} \mapsto \mathbf{!}_w$. For simplicity, we assume that an assignment has sufficient domain of definition for interpretation. Given a **BML**-model $\mathfrak{M} = \langle W, \preccurlyeq, \{M_w\}_{w\in W}\rangle$, the *satisfaction* of a formula A at $d \in D_w$ with ρ on $w \in W$, written $\mathfrak{M}, w, d \Vdash^\rho A$, is defined as

$$\mathfrak{M},\ w,\ d \Vdash^\rho A \qquad \iff \forall v \succcurlyeq w.\ \big(\mathfrak{M},\ v,\ d \vDash^\rho A\big),$$

where $\mathfrak{M}, w, d \vDash^\rho A$ is defined as follows:

$$\begin{aligned}
&\mathfrak{M},\ w,\ d \vDash^\rho p && \iff d \in V_w(p);\\
&\mathfrak{M},\ w,\ d \vDash^\rho A \to B && \iff \forall e \succeq_w d.\ \big(\mathfrak{M},\ w,\ e \Vdash^\rho A \implies \mathfrak{M},\ w,\ e \Vdash^\rho B\big);\\
&\mathfrak{M},\ w,\ d \vDash^\rho \Box^{\succeq\gamma}A && \iff \forall e \sqsupseteq_w d.\ \big(\rho(\gamma) \preceq_w e \implies \mathfrak{M},\ w,\ e \Vdash^\rho A\big);\\
&\mathfrak{M},\ w,\ d \vDash^\rho \forall\gamma_1 :\succeq \gamma_2.\ A && \iff \forall e \succeq_w \rho(\gamma_2).\ \big(\mathfrak{M},\ w,\ d \Vdash^{\rho\cdot[\gamma_1 \mapsto e]} A\big).
\end{aligned}$$

One should be noted that $\Vdash$ and $\vDash$ are defined mutually recursively.

As **BML**-model captures growth of **BML**-structure, it has two intuitionistic relations: $\preccurlyeq$ in **BML**-model and $\preceq$ in **BML**-structure. Hence, persistency for **BML**-model is stated in more elaborated manner.

▶ **Lemma 4.2** (Semantical Persistency). *Given a* **BML**-*model $\mathfrak{M}$ and suppose $w \preccurlyeq v$ and $d \preceq_v e$. If $\mathfrak{M}, w, d \Vdash^\rho A$, then $\mathfrak{M}, v, e \Vdash^\rho A$.*

Comparing this to Lemma 3.2, we can see that $\preccurlyeq$ has a correspondence to inclusion between contexts. Building upon this intuition, we build a canonical model for **BML** in the next section.

As we have seen in this section, the birelational Kripke structure for syntactic structure of programs does not necessarily correspond to Kripke semantics for its logical counterpart. Rather, its Kripke semantics is captured by growing structure of such syntactic structures.

5 Soundness and Completeness

We confirm the correspondence between the Kripke semantics and the proof system. In preparation, we introduce additional definitions for Kripke semantics. Unlike satisfaction of a formula, the accessibility of each transition relation is independent of d and interpreted as:

$$\mathfrak{M},\ w \Vdash^{\rho} \gamma_1 \trianglelefteq \gamma_2 \iff \rho(\gamma_1) \trianglelefteq \rho(\gamma_2). \qquad (\text{where } \trianglelefteq \in \{\preceq, \sqsubseteq\})$$

The interpretation of a context Γ is determined based on $\mathrm{pos}(\Gamma)$ as follows:

$$\begin{aligned}
\mathfrak{M},\ w \Vdash^{\rho} \varepsilon &\iff \text{always};\\
\mathfrak{M},\ w \Vdash^{\rho} \Gamma,\ A^{\gamma} &\iff \mathfrak{M},\ w \Vdash^{\rho} \Gamma,\ \ \mathfrak{M},\ w \Vdash^{\rho} \mathrm{pos}(\Gamma) \preceq \gamma,\ \text{ and } \mathfrak{M},\ w,\ \rho(\gamma) \Vdash^{\rho} A;\\
\mathfrak{M},\ w \Vdash^{\rho} \Gamma,\ \blacktriangleright^{\gamma_2 :\succeq \gamma_1} &\iff \mathfrak{M},\ w \Vdash^{\rho} \Gamma,\ \ \mathfrak{M},\ w \Vdash^{\rho} \mathrm{pos}(\Gamma) \sqsubseteq \gamma_2,\ \text{ and } \mathfrak{M},\ w \Vdash^{\rho} \gamma_1 \preceq \gamma_2;\\
\mathfrak{M},\ w \Vdash^{\rho} \Gamma,\ \blacktriangleleft^{\gamma} &\iff \mathfrak{M},\ w \Vdash^{\rho} \Gamma;\\
\mathfrak{M},\ w \Vdash^{\rho} \Gamma,\ \gamma_2 :\succeq \gamma_1 &\iff \mathfrak{M},\ w \Vdash^{\rho} \Gamma \text{ and } \mathfrak{M},\ w \Vdash^{\rho} \gamma_1 \preceq \gamma_2.
\end{aligned}$$

The *semantic consequence* is defined accordingly:

$$\begin{aligned}
\Gamma \Vdash A &\iff \forall \mathfrak{M},\ w,\ \rho.\ \big(\mathfrak{M},\ w \Vdash^{\rho} \Gamma \implies \mathfrak{M},\ w,\ \rho(\mathrm{pos}(\Gamma)) \Vdash^{\rho} A\big);\\
\Gamma \Vdash \gamma_1 \trianglelefteq \gamma_2 &\iff \forall \mathfrak{M},\ w,\ \rho.\ \big(\mathfrak{M},\ w \Vdash^{\rho} \Gamma \implies \mathfrak{M},\ w \Vdash^{\rho} \gamma_1 \trianglelefteq \gamma_2\big).
\end{aligned}$$
$$(\text{where } \trianglelefteq \in \{\preceq, \sqsubseteq\})$$

Now we can state soundness:

▶ **Theorem 5.1** (Kripke Soundness).

1. *If* $\Gamma \vdash A$, *then* $\Gamma \Vdash A$.
2. *If* $\Gamma \vdash \gamma_1 \preceq \gamma_2$, *then* $\Gamma \Vdash \gamma_1 \preceq \gamma_2$.
3. *If* $\Gamma \vdash \gamma_1 \sqsubseteq \gamma_2$, *then* $\Gamma \Vdash \gamma_1 \sqsubseteq \gamma_2$.

To prove completeness we use a *canonical-model* construction:

▶ **Definition 5.2** (Canonical Model). $\mathfrak{M}^{\mathsf{c}} = \langle W^{\mathsf{c}},\ \preccurlyeq^{\mathsf{c}},\ \{M^{\mathsf{c}}_{\Gamma}\}_{\Gamma \in W^{\mathsf{c}}}\rangle$ *is defined as follows:*

- W^{c} *is the set of all well-formed contexts;*
- $\Gamma \preccurlyeq^{\mathsf{c}} \Delta \iff \exists \Gamma'.\ (\Delta = \Gamma,\ \Gamma')$*;*
- $M^{\mathsf{c}}_{\Gamma} = \langle D^{\mathsf{c}}_{\Gamma},\ \preceq^{\mathsf{c}}_{\Gamma},\ \sqsubseteq^{\mathsf{c}}_{\Gamma},\ V^{\mathsf{c}}_{\Gamma},\ \mathbf{!}^{\mathsf{c}}_{\Gamma}\rangle$ *where*
  - $D^{\mathsf{c}}_{\Gamma} = \mathbf{Dom}_{\mathrm{C}}(\Gamma)$ *with* $\mathbf{!}^{\mathsf{c}}_{\Gamma} = \mathbf{!}$*;*
  - $\gamma_1 \preceq^{\mathsf{c}}_{\Gamma} \gamma_2 \iff \Gamma \vdash \gamma_1 \preceq \gamma_2$*;*
  - $\gamma_1 \sqsubseteq^{\mathsf{c}}_{\Gamma} \gamma_2 \iff \Gamma \vdash \gamma_1 \sqsubseteq \gamma_2$*;*
  - $\gamma \in V^{\mathsf{c}}_{\Gamma}(p) \iff \Gamma,\ \blacktriangleleft^{\mathbf{!}} \vdash \Box^{\succeq \gamma} p$*.*

▶ **Lemma 5.3.** $\mathfrak{M}^{\mathsf{c}}$ *is a* **BML***-model.*

This canonical model clarifies how the layered structure of **BML**-model corresponds to the structure of contexts in our proof system: each M^{c}_{Γ} models the structure formed by *classifiers*, whereas $\mathfrak{M}^{\mathsf{c}}$ models the structure formed by *contexts*. The relation $\preceq^{\mathsf{c}}_{\Gamma}$ in a **BML**-structure of the canonical model corresponds to the judgment $\Gamma \vdash \gamma_1 \preceq \gamma_2$ while the relation $\preccurlyeq$ corresponds to an order between contexts.

On stating truth lemma, it should be noted that there is a subtle difference between Kripke semantics and proof system. In semantics, the truth of a formula is always defined at each point of a model, whereas in syntax, only its validity at $\mathrm{pos}(\Gamma)$ is assertible under Γ; in this respect, **BML** differs from ordinary labelled proof systems (cf. e.g., [25, 30]).

$$\begin{array}{lll}
\textbf{Types} & A,\ B & ::= p \mid A \to B \mid \Box^{\succeq\gamma} A \mid \forall\gamma_1 :\succeq \gamma_2.\ A \\
\textbf{Contexts} & \Gamma,\ \Delta & ::= \varepsilon \mid \Gamma,\ x :^{\gamma} A \mid \Gamma,\ \blacktriangleright^{\gamma_1 :\succeq \gamma_2} \mid \Gamma,\ \blacktriangleleft^{\gamma} \mid \Gamma,\ \gamma_1 :\succeq \gamma_2 \\
\textbf{Terms} & M,\ N & ::= x \mid \lambda x :^{\gamma} A.\, M \mid M_1 M_2 \mid \mathbf{quo}\{^{\gamma_1 :\succeq \gamma_2} M\} \mid \mathbf{unq}\{^{\gamma} M\} \mid \lambda\gamma_1 :\succeq \gamma_2.\ M \mid M\gamma
\end{array}$$

$$\textbf{Var}\quad \dfrac{x :^{\gamma_1} A \in \Gamma^{\gamma_2} \qquad \Gamma^{\gamma_2} \vdash \gamma_1 \preceq \gamma_2}{\Gamma^{\gamma_2} \vdash x : A}
\qquad
\textbf{$\to$-I}\quad \dfrac{\Gamma,\ x :^{\gamma} A_1 \vdash M_1 : A_2 \qquad \gamma_1 \notin \mathbf{FC}(A_2)}{\Gamma \vdash \lambda x :^{\gamma} A_1.\, M_1 : A_1 \to A_2}
\qquad
\textbf{$\to$-E}\quad \dfrac{\Gamma \vdash M_1 : A_1 \to A_2 \qquad \Gamma \vdash M_2 : A_1}{\Gamma \vdash M_1 M_2 : A_2}$$

$$\textbf{$\Box$-I}\quad \dfrac{\Gamma,\ \blacktriangleright^{\gamma_1 :\succeq \gamma_2} \vdash M : A \qquad \gamma_1 \notin \mathbf{FC}(A)}{\Gamma \vdash \mathbf{quo}\{^{\gamma_1 :\succeq \gamma_2} M\} : \Box^{\succeq \gamma_2} A}
\qquad
\textbf{$\Box$-E}\quad \dfrac{\Gamma^{\gamma_1},\ \blacktriangleleft^{\gamma_2} \vdash M : \Box^{\succeq \gamma_3} A \qquad \Gamma^{\gamma_1} \vdash \gamma_3 \preceq \gamma_1}{\Gamma^{\gamma_1} \vdash \mathbf{unq}\{^{\gamma_2} M\} : A}$$

$$\textbf{$\forall$-I}\quad \dfrac{\Gamma,\ \gamma_1 :\succeq \gamma_2 \vdash M : A}{\Gamma \vdash \lambda\gamma_1 :\succeq \gamma_2.\ M : \forall\gamma_1 :\succeq \gamma_2.\ A}
\qquad
\textbf{$\forall$-E}\quad \dfrac{\Gamma \vdash M : \forall\gamma_1 :\succeq \gamma_2.\ A \qquad \Gamma \vdash \gamma_2 \preceq \gamma_3}{\Gamma \vdash M\gamma_3 : A[\gamma_1 \coloneqq \gamma_3]}$$

Figure 7 Definitions and typing judgments for our lambda-calculus. We also have rules for $\vdash \Gamma :\ \mathbf{ctx}$, $\Gamma \vdash A :\ \mathbf{type}$, $\Gamma \vdash \gamma_1 \preceq \gamma_2$ and $\Gamma \vdash \gamma_1 \sqsubseteq \gamma_2$. We omit them as they are almost identical to those in the natural-deduction system.

The restriction, however, does not pose a problem for establishing their correspondence, because $\Gamma, \blacktriangleleft^{!} \Vdash \Box^{\succeq\gamma} A$ can be used instead to represent the validity of A at γ under Γ. Here, the bounded modality expresses monotonicity, analogous to the **S4** modality in the Gödel–McKinsey–Tarski translation.

The *canonical* Γ*-assignment* $\rho^{\mathsf{c}}_{\Gamma}$ is an assignment that maps each $\gamma \in \mathbf{Dom}_{\mathrm{C}}(\Gamma)$ to itself, and we write $\Gamma, \gamma \Vdash^{\mathsf{c}} A$ if $\mathfrak{M}^{\mathsf{c}}, \Gamma, \gamma \Vdash^{\rho^{\mathsf{c}}_{\Gamma}} A$. The *truth lemma* is states in the following form:

▶ **Lemma 5.4** (Truth Lemma). $\Gamma, \gamma \Vdash^{\mathsf{c}} A \iff \Gamma, \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma} A$.

Finally, we show Kripke completeness:

▶ **Theorem 5.5** (Kripke Completeness).

1. *If* $\Gamma \Vdash A$, *then* $\Gamma \vdash A$.
2. *If* $\Gamma \Vdash \gamma_1 \preceq \gamma_2$, *then* $\Gamma \vdash \gamma_1 \preceq \gamma_2$.
3. *If* $\Gamma \Vdash \gamma_1 \sqsubseteq \gamma_2$, *then* $\Gamma \vdash \gamma_1 \sqsubseteq \gamma_2$.

6 Lambda-Calculus and Metatheory

Under the Curry–Howard isomorphism [9,31], we can consider a typed lambda-calculus that corresponds to our natural-deduction system. One can find the relevant definitions and typing rules in Figure 7. In the resulting calculus, one can observe that terms for →-I, □-I, □-E, ∀-I correspond to items in context; thus, context structure and its **BML**-structure capture the syntactic structure of lambda-terms. Particularly, terms for □-I and □-E appear as quasi-quotation constructs, which is commonly observed in modal lambda-calculi [5–7,18]. As an example, Figure 8 shows how our calculus represents the example in Figure 1c, and its corresponding **BML**-structure, which demonstrates that **BML** allows typing programs that were not typed under the traditional modal language.

In this section, we study the metatheory of the calculus including subject reduction and strong normalization. These results ensure that proof normalization is well behaved and that every well-typed term admits a normal form, providing a computational justification for our proof-theoretic development.

```
(*):γ0 int→int→int, 10:γ1 int ⊢
  let sqr:γ2 int→int = λx:γ3 int. x * x in
  let code:γ4 □⪰γ2 int = quo{γ5:⪰γ2 sqr 10 } in
  unq{γ4 code }
  : int
```

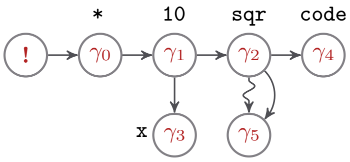


Figure 8 The example in Figure 1c written in our calculus, and its corresponding **BML**-structure. As we lack constants, we introduce `*` and `10` as variables. `let`-expression is a shorthand for λ-abstraction and application.

6.1 Logical Harmony and Reduction Semantics

In preparation for discussion on proof reduction, we introduce two meta operations and confirm their properties. Classifier substitution $(-)[\gamma_1 := \gamma_2]$ operates on classifiers, contexts, types and terms, substituting free occurrences of γ_1 with γ_2. Variable substitution $(-)[\gamma_1 := \gamma_2, x := M]$ operates on terms, substituting free occurrences of γ_1 and x with γ_2 and M, respectively. Then, we can confirm that the following lemmas hold.

▶ **Lemma 6.1** (Variable Substitution). *If* $\Gamma_1^{\gamma_1}, x :^{\gamma_2} A, \Gamma_2 \vdash M_1 : B$ *and* $\Gamma_1 \vdash M_2 : A$, *then* $\Gamma_1, \Gamma_2[\gamma_2 := \gamma_1] \vdash M_1[\gamma_2 := \gamma_1, x := M_2] : B[\gamma_2 := \gamma_1]$.

▶ **Lemma 6.2** (Rebasing). *If* $\Gamma_1^{\gamma_1}, \blacktriangleleft^{\gamma_2}, \blacktriangleright^{\gamma_3 :\succeq \gamma_4}, \Gamma_2 \vdash M_1 : A$ *and* $\Gamma_1 \vdash \gamma_4 \preceq \gamma_1$, *then* $\Gamma_1, \Gamma_2[\gamma_3 := \gamma_1] \vdash M[\gamma_3 := \gamma_1] : A[\gamma_3 := \gamma_1]$

▶ **Lemma 6.3** (Classifier Substitution). *If* $\Gamma_1, \gamma_1 :\succeq \gamma_2, \Gamma_2 \vdash M : A$ *and* $\Gamma_1 \vdash \gamma_2 \preceq \gamma_3$, *then* $\Gamma_1, \Gamma_2[\gamma_1 := \gamma_3] \vdash M[\gamma_1 := \gamma_3] : A[\gamma_1 := \gamma_3]$.

The variable substitution and classifier substitution lemmas are standard in typed lambda-calculi. In particular, these substitutions are based on the standard notion of substitution for well-typed terms and are extended to classifiers in a natural way. On the other hand, the rebasing lemma is specific to our calculus, which states that ▶ cancels ◀, which globally changes the corresponding **BML**-structure.

Thereafter, we can confirm that the introduction and elimination rules in **BML** are well balanced with respect to local soundness and local completeness [27]. For the sake of space, we skim local soundness/completeness patterns in the following statements.

▶ **Lemma 6.4** (Local Soundness Patterns).

1. $\Gamma^{\gamma_1} \vdash (\lambda x :^{\gamma_2} A.\, M)N : B \implies \Gamma \vdash M[\gamma_2 := \gamma_1, x := N] : B.$
2. $\Gamma^{\gamma_1} \vdash \mathbf{unq}\{^{\gamma_2}\mathbf{quo}\{^{\gamma_3:\succeq\gamma_4} M\}\} : A \implies \Gamma \vdash M[\gamma_3 := \gamma_1] : A.$
3. $\Gamma \vdash (\lambda \gamma_1 :\succeq \gamma_2.\, M)\gamma_3 : A \implies \Gamma \vdash M[\gamma_1 := \gamma_3] : A.$

▶ **Lemma 6.5** (Local Completeness Patterns). *(δ is taken freshly)*

1. $\Gamma \vdash M : A \to B \implies \Gamma \vdash \lambda x :^{\delta} A.\, (Mx) : A \to B.$
2. $\Gamma^{\gamma_1} \vdash M : \Box^{\succeq\gamma_2} A \implies \Gamma \vdash \mathbf{quo}\{^{\delta:\succeq\gamma_2}\mathbf{unq}\{^{\gamma_1} M\}\} : \Box^{\succeq\gamma_2} A.$
3. $\Gamma \vdash M : \forall\gamma_1 :\succeq \gamma_2.\ A \implies \Gamma \vdash \lambda\delta :\succeq \gamma_2.\ (M\delta) : \forall\gamma_1 :\succeq \gamma_2.\ A.$

Local soundness and completeness patterns can be regarded as β-reduction and η-expansion, respectively. We define β-reduction on raw terms, notated $M_1 \Rightarrow_\beta^\gamma M_2$. Unlike standard definitions of β-reduction, that in our calculus is a family of relations indexed by classifiers. This classifier stands for the position where the reduction takes place.

▶ **Definition 6.6** (β-Reduction). *$M_1 \Rightarrow^{\gamma}_{\beta} M_2$ is defined to satisfy following rules (along with compatibility rules).*

$$\begin{aligned}(\lambda x :^{\gamma_2} A.\, M)N &\Rightarrow^{\gamma_1}_{\beta} M[\gamma_2 \coloneqq \gamma_1,\, x \coloneqq N]\\ \mathbf{unq}\{^{\gamma_2}\mathbf{quo}\{^{\gamma_3 :\succeq \gamma_4} M\}\} &\Rightarrow^{\gamma_1}_{\beta} M[\gamma_3 \coloneqq \gamma_1]\\ (\lambda \gamma_2 :\succeq \gamma_3.\, M)\gamma_4 &\Rightarrow^{\gamma_1}_{\beta} M[\gamma_2 \coloneqq \gamma_4]\end{aligned}$$

We write $\Rightarrow^{\gamma}_{\beta *}$ for the reflexive and transitive closure of $\Rightarrow^{\gamma}_{\beta}$.

Note that the compatibility rules need to maintain positions for terms that move positions. We can confirm that β-reduction preserves typeability.

▶ **Theorem 6.7** (Subject Reduction). *If $\Gamma^{\gamma} \vdash M_1 : A$ and $M_1 \Rightarrow^{\gamma}_{\beta} M_2$, then $\Gamma \vdash M_2 : A$.*

6.2 Metatheory

β-reduction behaves well in the following sense:

▶ **Theorem 6.8** (Strong Normalization). *If $\Gamma^{\gamma} \vdash M : A$, then M is strongly normalizing with respect to $\Rightarrow^{\gamma}_{\beta}$.*

▶ **Theorem 6.9** (Confluence). *If $\Gamma^{\gamma} \vdash M_1 : A$, $M_1 \Rightarrow^{\gamma}_{\beta *} M_2$ and $M_1 \Rightarrow^{\gamma}_{\beta *} M_3$, then there exists M_4 such that $M_2 \Rightarrow^{\gamma}_{\beta *} M_4$ and $M_3 \Rightarrow^{\gamma}_{\beta *} M_4$.*

▶ **Corollary 6.10** (Uniqueness of β-Normal Form). *If $\Gamma^{\gamma} \vdash M_1 : A$, then M_1 has a unique normal form with respect to $\Rightarrow^{\gamma}_{\beta}$.*

In order to confirm strong normalization, it suffices to reduce it to strong normalization of simply typed lambda-calculus, which is a common technique in modal lambda-calculi [6, 18]. Instead, we proved it directly by using the method of *reducibility*, which can be found in Appendix. Finally, we confirm canonicity and the subformula property.

▶ **Definition 6.11.**

1. *A term is said to be* canonical *if its outermost term-former is for an introduction rule, and is said to be* neutral *otherwise.*
2. *A* subformula *of a formula is a literal subexpression with some classifier maybe renamed.*

▶ **Theorem 6.12** (Canonicity). *If a term is well-typed, closed regarding term variable, and β-normal, then it is canonical.*

▶ **Theorem 6.13** (Subformula Property). *Suppose $\Gamma^{\gamma} \vdash M : A$. If M is normal with respect to $\Rightarrow^{\gamma}_{\beta}$, then any subterm of M satisfies at least one of the following:*

1. *Its type is a subformula of A;*
2. *Its type is a subformula of B for some $x :^{\delta} B \in \Gamma$.*

7 BML as a Generalization of S4

We designed **BML** as a generalization of **S4**, and we formally confirm this fact in this section. We interpret the S4 modal operator $\Box$ as a bounded modal operator $\Box^{\succeq !}$, with respect to empty resource. For formal comparison, we consider a restricted language of **BML** and a language of **S4**:

$$\mathcal{L}_{!} \ni A,\, B ::= p \mid A \to B \mid \Box^{\succeq !} A; \qquad\qquad \mathcal{L}_{\Box} \ni A,\, B ::= p \mid A \to B \mid \Box A.$$

And we define $|{-}| : \mathcal{L}_{!} \to \mathcal{L}_{\Box}$ and $(-)^{\succeq !} : \mathcal{L}_{\Box} \to \mathcal{L}_{!}$ as translations that swap $\Box^{\succeq !}$ and $\Box$. In this section, we dig into the correspondence between **BML** and **S4** via these translations, both from semantical and proof-theoretic perspectives.

7.1 Semantical Comparison

Alechina et al. [2] introduced a birelational model for **CS4**, a *constructive* variant of **S4**:

▶ **Definition 7.1** (CS4-Model [2]). *A* **CS4**-model[3] *is a quadruple* $\langle W, \preceq, R, V\rangle$*, where*

- $\langle W, \preceq\rangle$ *is a nonempty preordered set,*
- R *is a preorder on* W *with condition:*
  - Left-persistency: $(R\,;\preceq) \subseteq (\preceq\,; R)$*, and*
- V *assigns each atom* p *to an upward-closed subset of* W*.*

Then we define satisfaction of a modal operator $M, w \vDash_{\mathbf{CS4}} \Box A$ as $\forall v.\ w \preceq v \Rightarrow \forall u.\ v\ R\ u \Rightarrow M, u \vDash_{\mathbf{CS4}} A$. From semantical perspective, we argue that there is a mutual translation between **BML** models and **CS4** models. From **CS4** models to **BML** models, we take steps as described below, which we call *one-point-model construction*:

▶ **Lemma 7.2** (Stabilization). *Given a* **CS4***-model* $M = \langle W, \preceq, R, V\rangle$*. Define* $\sqsubseteq$ *as* $(\preceq\,; R)$*. Then* $M^* = \langle W, \preceq, \sqsubseteq, V\rangle$ *is a stable* **CS4***-model.*

▶ **Lemma 7.3** (Root-Extension). *Given a stable* **CS4***-model* $M = \langle W, \preceq, \sqsubseteq, V\rangle$*. Define* $M_{\mathbf{!}} = \langle W_{\mathbf{!}}, \preceq_{\mathbf{!}}, \sqsubseteq_{\mathbf{!}}, V_{\mathbf{!}}\rangle$ *as follows:*

- $W_{\mathbf{!}} = W \amalg \{\mathbf{!}\}$;
- $w \preceq_{\mathbf{!}} v \iff w = \mathbf{!}$ *or* $w \preceq v$;
- $w \sqsubseteq_{\mathbf{!}} v \iff w = \mathbf{!}$ *or* $w \sqsubseteq v$;
- $w \in V_{\mathbf{!}}(p) \iff V(p) = W$ *if* $w = \mathbf{!}$*, and* $w \in V(p)$ *otherwise.*

Then $M_{\mathbf{!}}$ *is a stable* **CS4***-model with a root* $\mathbf{!}$*, namely, a* **BML***-structure.*

▶ **Lemma 7.4** (One-Point Model). *Given a* **BML***-structure* M*. Define* M_* *as*

$$\langle \{*\}, \{\langle *, *\rangle\}, \{* \mapsto M\}\rangle.$$

Then M_* *is a* **BML***-model.*

By combining these steps, we can construct a **BML**-model from a **CS4**-model. We can confirm that such a **BML**-model behaves equivalently to the original **CS4**-model:

▶ **Theorem 7.5.** *Given a* **CS4***-model* $M = \langle W, \preceq, R, V\rangle$*. Define* $\mathfrak{M}$ *as* $(M^*)_{\mathbf{!}*}$ *and a* $*$*-assignment* $\mathbf{!}$ *for* $\mathfrak{M}$ *as* $\mathbf{!} \mapsto \mathbf{!}$*. Then* $\mathfrak{M}$ *is a* **BML***-model, and for any* $A \in \mathcal{L}_\Box$*, the following are equivalent:*

- $M, w \vDash_{\mathbf{CS4}} A$;
- $\mathfrak{M}, *, w \Vdash^{\mathbf{!}} (A)^{\succeq \mathbf{!}}$.

Conversely, we construct a **CS4**-model from a **BML**-model via *flattening*.

▶ **Definition 7.6** (Flattening). *Let* $\mathfrak{M} = \langle W, \preccurlyeq, \{M_w\}_{w \in W}\rangle$ *be a* **BML***-model, with each* M_w *as* $\langle D_w, \preceq_w, \sqsubseteq_w, V_w, \mathbf{!}_w\rangle$*. Then a* **CS4***-model* $\mathfrak{M}_+ = \langle W_+, \preceq_+, R_+, V_+\rangle$ *is defined as follows:*

- $W_+ = \sum_{w \in W} D_w$;
- $\langle w, d\rangle \preceq_+ \langle w', d'\rangle \iff w \preccurlyeq w'$ *and* $d \preceq_{w'} d'$;
- $\langle w, d\rangle\, R_+\, \langle w', d'\rangle \iff w = w'$ *and* $d \sqsubseteq_{w'} d'$;
- $\langle w, d\rangle \in V_+(p) \iff d \in V_w(p)$.

---

[3] For simplicity, we omit *fallible worlds* from the definition because $\bot$ is not considered in this paper, but our model can be extended to having them as well.

▶ **Lemma 7.7.** *$\mathfrak{M}_+$ is a* **CS4***-model.*

▶ **Theorem 7.8.** *Given a* **BML***-model $\mathfrak{M}$. For any $A \in \mathcal{L}_{\mathbf{!}}$, the following are equivalent:*

- $\mathfrak{M},\, w,\, d \Vdash^{\rho} A$;
- $\mathfrak{M}_+,\, \langle w,\, d\rangle \vDash_{\mathbf{CS4}} |A|$.

These theorems indicate that the two constructions, namely $(M^*)_{!*}$ and $\mathfrak{M}_+$, are pseudo-inverse operations that preserve satisfaction, leading to the following characterization:

▶ **Theorem 7.9.** *The $\mathcal{L}_{\Box}$-fragment of* **CS4** *is isomorphic to the $\mathcal{L}_{\mathbf{!}}$-fragment of* **BML** *up to logical equivalence.*

7.2 Proof-Theoretic Comparison

As discussed in Section 3, our proof system can be regarded as an extension of Kripke/Fitch-style modal proof systems [5,7,18]. Leveraging this correspondence, we can translate proofs in their proof systems to ours. Particularly, we prove that we can translate proofs in Kripke-style proof system for **S4** [7] into proofs in our proof system. We get the following result:

▶ **Theorem 7.10.** *If $\varepsilon \vdash_{\mathbf{S4}} A$, then $\varepsilon \vdash (A)^{\succeq \mathbf{!}}$.*

According to Theorem 7.9, the converse direction is also expected to hold.

8 Related Work

The key charasteristics of **BML** can be summarized in two aspects. First, **BML** introduces a first-order language with classifiers to reason about the persistent nature of intuitionistic relations. Second, it integrates modal and first-order languages via bounded modality, which explicitly captures interactions between modal and intuitionistic transitions.

For the first aspect, closely related ideas can be found in labelled proof systems for intuitionistic logic [4,8,17], where labels are used to represent worlds and enable proof-theoretic reasoning guided by Kripke semantics. While this line of work has been extensively studied from a logical perspective, its computational interpretation has received relatively little attention. To the best of our knowledge, the proposal by Reed and Pfenning [28] is one of the few works that develop a labelled proof system for intuitionistic logic together with a corresponding lambda-calculus. Their work provides a logical foundation for control operators based on intuitionistic logic. In their system, labels represent binding positions and are used to detect illegal uses of continuations. This use of labels is conceptually close to our interpretation of classifiers as variable scopes, although the technical setting and intended applications are different.

Related ideas also appear in programming language theory, where variable scopes or resource availability are made explicit at the type level. A notable example is region-based memory management [10,36] and lifetime-based systems [11], which are used for the static management of regions in which objects are alive. Some of these systems formalize inclusion relations between regions or lifetimes, which is similar to our formulation of classifiers. Another closely related line of work is refined environment classifiers [13,15,26], which annotate scope information on code types to ensure safe code generation in the presence of computational effects. This line of work inspired our development of **BML** as a logical reconstruction of their approach. However, there is no direct correspondence between their type systems and our calculus, nor is such a correspondence our goal; for example, their calculi do not support programs like Figure 8.

For the second aspect, several proposals have explored type systems that represent resource isolation in a more flexible manner. A prominent example is Contextual Modal Logic (**CML**) [22, 24], which extends the modal operator of **CS4** with explicit contexts. In **CML**, the contextual modality $[\Gamma \vdash A]$ expresses that A is valid under the assumptions Γ. From this perspective, contexts can be understood as a way of representing persistent resources, which bears some similarity to bounded modality. However, the technical setting and design goals of **CML** are quite different from ours. In particular, **CML** does not explicitly model interactions between intuitionistic and modal transitions, and contextual modality does not provide a notion of persistency across modal boundaries. As a result, **CML** does not directly account for programs that involve subtle forms of resource control. For example, it does not type programs like Figure 1c, reflecting a difference in design goals rather than a limitation of the system.

9 Conclusion and Future Work

In this paper, we explore an experimental extension of constructive modal logic motivated by a birelational Kripke-structural view of syntactic resource control in programming languages. We introduced **BML** as a modal logic that captures the interaction between intuitionistic and modal transitions in birelational Kripke structures. We formalized **BML** by presenting a natural-deduction proof system and a Kripke semantics, and established their correspondence. We further provided a computational interpretation of **BML** via a corresponding modal lambda-calculus and studied its fundamental metatheoretic properties. Finally, we clarified that **BML** can be viewed as a generalization of **S4**. Taken together, these results demonstrate that **BML** is a well-disciplined logic. They support the idea that birelational Kripke structures provide a viable basis for exploring fine-grained resource control in programming languages.

From a logical perspective, **BML** integrates a modal language and a first-order language, connected by bounded modality, which serves as an explicit interface between modal and intuitionistic transitions. This duality is reflected uniformly in both the proof system and the Kripke semantics. From a computational perspective, **BML** provides a language for describing interactions between variable scopes and resource isolation. This suggests that **BML** may offer typing disciplines for programs that fall outside the scope of traditional modal lambda-calculi. In this paper, however, we focused on establishing the theoretical foundations of **BML** and its computational counterpart.

Whether **BML** can serve as a foundation for practical programming applications, however, remains an open question. As discussed in Section 1, our primary motivation lies in applications to multi-stage programming. In this setting, properties such as time-ordered normalization [6, 39] are widely regarded as desirable. Addressing this question will require a more detailed analysis of the operational behavior of the corresponding modal lambda-calculus.

Beyond practical applications, an interesting direction for future work is to explore bounded extensions of modal logics other than **S4**. For logics such as **K**, **K4**, and **T**, bounded variants can be obtained by relatively minor modifications of our proof system, in particular by adjusting the rules for the modal relation $\sqsubseteq$. In contrast, for modal logics such as **S5** [23] and **GL** [12], a straightforward adaptation appears insufficient, and more substantial proof-theoretic extensions may be required. We leave this investigation to future work.

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$\boxed{\vdash \Gamma : \mathbf{ctx}}$

$$\textbf{WF-Emp}\ \frac{}{\vdash \varepsilon : \mathbf{ctx}} \qquad \textbf{WF-Hyp}\ \frac{\vdash \Gamma : \mathbf{ctx} \qquad \Gamma \vdash A : \mathbf{prop} \qquad \gamma \notin \mathbf{Dom}_{\mathrm{C}}(\Gamma)}{\vdash \Gamma,\, A^{\gamma} : \mathbf{ctx}}$$

$$\textbf{WF-}\blacktriangleright\ \frac{\vdash \Gamma : \mathbf{ctx} \qquad \gamma_1 \notin \mathbf{Dom}_{\mathrm{C}}(\Gamma) \qquad \gamma_2 \in \mathbf{Dom}_{\mathrm{C}}(\Gamma)}{\vdash \Gamma,\, \blacktriangleright^{\gamma_1 :\succeq \gamma_2} : \mathbf{ctx}} \qquad \textbf{WF-}\blacktriangleleft\ \frac{\vdash \Gamma^{\gamma} : \mathbf{ctx} \qquad \Gamma^{\gamma} \vdash \delta \sqsubseteq \gamma}{\vdash \Gamma^{\gamma},\, \blacktriangleleft^{\delta} : \mathbf{ctx}}$$

$$\textbf{WF-cls}\ \frac{\vdash \Gamma : \mathbf{ctx} \qquad \gamma_1 \notin \mathbf{Dom}_{\mathrm{C}}(\Gamma) \qquad \gamma_2 \in \mathbf{Dom}_{\mathrm{C}}(\Gamma)}{\vdash \Gamma,\, \gamma_1 :\succeq \gamma_2 : \mathbf{ctx}}$$

$\boxed{\Gamma \vdash A : \mathbf{prop}}$

$$\textbf{WF-atom}\ \frac{\vdash \Gamma : \mathbf{ctx}}{\Gamma \vdash p : \mathbf{prop}} \qquad \textbf{WF-}\to\ \frac{\Gamma \vdash A : \mathbf{prop} \qquad \Gamma \vdash B : \mathbf{prop}}{\Gamma \vdash A \to B : \mathbf{prop}}$$

$$\textbf{WF-}\Box\ \frac{\Gamma \vdash A : \mathbf{prop} \qquad \gamma \in \mathbf{Dom}_{\mathrm{C}}(\Gamma)}{\Gamma \vdash \Box^{\succeq \gamma} A : \mathbf{prop}} \qquad \textbf{WF-}\forall\ \frac{\Gamma,\, \gamma_1 :\succeq \gamma_2 \vdash A : \mathbf{prop}}{\Gamma \vdash \forall \gamma_1 :\succeq \gamma_2 .\ A : \mathbf{prop}}$$

Figure 9 Derivation Rules for Well-Formedness Judgments

A Full Definitions for Section 3

▶ **Definition A.1.** $\mathbf{FC}(A)$ *represents a set of free classifiers in* A*.*

$$\begin{aligned} \mathbf{FC}(p) &= \emptyset \\ \mathbf{FC}(A \to B) &= \mathbf{FC}(A) \cup \mathbf{FC}(B) \\ \mathbf{FC}(\Box^{\succeq \gamma} A) &= \mathbf{FC}(A) \cup \{\gamma\} \\ \mathbf{FC}(\forall \gamma_1 :\succeq \gamma_2 .\ A) &= (\mathbf{FC}(A) - \{\gamma_1\}) \cup \{\gamma_2\} \end{aligned}$$

▶ **Definition A.2.** $\mathbf{Dom}_{\mathrm{C}}(\Gamma)$ *represents a set of classifiers and variables declared in* Γ*.*

$$\begin{aligned} \mathbf{Dom}_{\mathrm{C}}(\varepsilon) &= \{!\} \\ \mathbf{Dom}_{\mathrm{C}}(\Gamma,\, A^{\gamma}) &= \mathbf{Dom}_{\mathrm{C}}(\Gamma) \cup \{\gamma\} \\ \mathbf{Dom}_{\mathrm{C}}(\Gamma,\, \blacktriangleright^{\gamma_1 :\succeq \gamma_2}) &= \mathbf{Dom}_{\mathrm{C}}(\Gamma) \cup \{\gamma_1\} \\ \mathbf{Dom}_{\mathrm{C}}(\Gamma,\, \blacktriangleleft^{\gamma}) &= \mathbf{Dom}_{\mathrm{C}}(\Gamma) \\ \mathbf{Dom}_{\mathrm{C}}(\Gamma,\, \gamma_1 :\succeq \gamma_2) &= \mathbf{Dom}_{\mathrm{C}}(\Gamma) \cup \{\gamma_1\} \end{aligned}$$

▶ **Definition A.3.** *A well-formed context judgment* $\vdash \Gamma : \mathbf{ctx}$ *and well-formed type judgment* $\Gamma \vdash A : \mathbf{prop}$ *are derived by rules listed in Figure 9*

▶ **Definition A.4.** *A classifier substitution* $(-)[\gamma_1 := \gamma_2]$ *is a meta operation on classifiers and types, which replaces free occurrences of* γ_1 *with* γ_2*.*

$$\gamma_1[\gamma_2 \coloneqq \gamma_3] = \begin{cases} \gamma_3 & \textit{if } \gamma_1 = \gamma_2 \\ \gamma_1 & \textit{otherwise} \end{cases}$$

$$\begin{aligned} p[\gamma_1 \coloneqq \gamma_2] &= p \\ (A \to B)[\gamma_1 \coloneqq \gamma_2] &= A[\gamma_1 \coloneqq \gamma_2] \to B[\gamma_1 \coloneqq \gamma_2] \\ \Box^{\succeq \gamma_1} A[\gamma_2 \coloneqq \gamma_3] &= \Box^{\succeq \gamma_1[\gamma_2 \coloneqq \gamma_3]} A[\gamma_2 \coloneqq \gamma_3] \\ (\forall \gamma_1 :\succeq \gamma_2.\ A)[\gamma_3 \coloneqq \gamma_4] &= \forall \gamma_1 :\succeq \gamma_2[\gamma_3 \coloneqq \gamma_4].\ A[\gamma_3 \coloneqq \gamma_4] \\ & \quad \textit{where } \gamma_1 \notin \{\gamma_3, \gamma_4\} \end{aligned}$$

$$\begin{aligned} \varepsilon[\gamma_1 \coloneqq \gamma_2] &= \varepsilon \\ (A^{\gamma_1}, \Gamma)[\gamma_2 \coloneqq \gamma_3] &= A[\gamma_2 \coloneqq \gamma_3]^{\gamma_1}, \Gamma[\gamma_2 \coloneqq \gamma_3] \\ & \quad \textit{where } \gamma_1 \notin \{\gamma_2, \gamma_3\} \\ (\blacktriangleright^{\gamma_1 :\succeq \gamma_2}, \Gamma)[\gamma_3 \coloneqq \gamma_4] &= \blacktriangleright^{\gamma_1 :\succeq \gamma_2[\gamma_3 \coloneqq \gamma_4]}, \Gamma[\gamma_3 \coloneqq \gamma_4] \\ & \quad \textit{where } \gamma_1 \notin \{\gamma_3, \gamma_4\} \\ (\blacktriangleleft^{\gamma_1}, \Gamma)[\gamma_2 \coloneqq \gamma_3] &= \blacktriangleleft^{\gamma_1[\gamma_2 \coloneqq \gamma_3]}, \Gamma[\gamma_2 \coloneqq \gamma_3] \\ (\gamma_1 :\succeq \gamma_2, \Gamma)[\gamma_3 \coloneqq \gamma_4] &= \gamma_1 :\succeq \gamma_2[\gamma_3 \coloneqq \gamma_4], \Gamma[\gamma_3 \coloneqq \gamma_4] \\ & \quad \textit{where } \gamma_1 \notin \{\gamma_3, \gamma_4\} \end{aligned}$$

B Proofs for Section 4

▶ **Lemma 4.2** (Semantical Persistency (On page 8))**.** *Given a* **BML**-*model* $\mathfrak{M}$ *and suppose* $w \preccurlyeq v$ *and* $d \preceq_v e$*. If* $\mathfrak{M}, w, d \Vdash^\rho A$*, then* $\mathfrak{M}, v, e \Vdash^\rho A$*.*

Proof. It suffices to check that for all $d' \succeq_w d$, if $w, d \vDash^\rho A$, then $w, d' \vDash^\rho A$. We proceed by induction on A.

▷ Case ($A \equiv p$). Follows from that $V_w(p)$ is upward-closed.

▷ Case ($A \equiv B \to C$). By definition.

▷ Case ($A \equiv \Box^{\succeq \gamma} B$). Suppose $w, d \vDash^\rho \Box^{\succeq \gamma} B$ and $d \preceq_w d'$. Take $e \sqsupseteq_w d'$ such that $e \succeq_w \rho(\gamma)$. By left-stability we have $d \sqsubseteq_w e$, so that $w, e \Vdash^\rho B$ since $w, d \vDash^\rho \Box^{\succeq \gamma} B$, which implies $w, e \vDash^\rho \Box^{\succeq \gamma} B$.

▷ Case ($A \equiv \forall \gamma_2 :\succeq \gamma_1.\ B$). Immediate from the IH. ◀

C Proofs for Section 5

▶ **Lemma C.1.** *Suppose* $w \preccurlyeq v$ *and* $d \preceq_v e$*.*

1. *If* $w \Vdash^\rho \gamma_1 \preceq \gamma_2$*, then* $v \Vdash^\rho \gamma_1 \preceq \gamma_2$*.*
2. *If* $w \Vdash^\rho \gamma_1 \sqsubseteq \gamma_2$*, then* $v \Vdash^\rho \gamma_1 \sqsubseteq \gamma_2$*.*

Proof. Straightforward. ◀

▶ **Corollary C.2.** *If* $w \preccurlyeq v$ *and* $w \Vdash^\rho \Gamma$*, then* $v \Vdash^\rho \Gamma$*.*

▶ **Lemma C.3** (Substitution). $w,\, d \Vdash^{\rho\cdot[\gamma_2 \mapsto \rho(\gamma_1)]} A \iff w,\, d \Vdash^{\rho} A[\gamma_2 := \gamma_1]$.

Proof. By induction on A. ◀

▶ **Theorem 5.1** (Kripke Soundness (On page 9)).

1. *If* $\Gamma \vdash A$, *then* $\Gamma \Vdash A$.
2. *If* $\Gamma \vdash \gamma_1 \preceq \gamma_2$, *then* $\Gamma \Vdash \gamma_1 \preceq \gamma_2$.
3. *If* $\Gamma \vdash \gamma_1 \sqsubseteq \gamma_2$, *then* $\Gamma \Vdash \gamma_1 \sqsubseteq \gamma_2$.

Proof. (2) and (3) are mostly straightforward, so we show (1) here. We proceed by induction on derivation.

Suppose $\Gamma \vdash A$. Assuming $w \Vdash^{\rho} \Gamma$, we show $w,\, \mathrm{pos}(\Gamma) \Vdash^{\rho} A$. We analyze the last rule of the derivation:

▷ Case (Var). Assume

$$\frac{x :^{\gamma} A \in \Gamma \qquad \Gamma \vdash \gamma \preceq \mathrm{pos}(\Gamma)}{\Gamma \vdash A}$$

By assumption we have $w,\, \rho(\gamma) \Vdash^{\rho} A$, and from (2), $\rho(\gamma) \preceq_w \rho(\mathrm{pos}(\Gamma))$ holds. By Lemma 4.2 we see $w,\, \rho(\mathrm{pos}(\Gamma)) \Vdash^{\rho} A$.

▷ Case (→-I). Assume

$$\frac{\Gamma, x :^{\gamma} B \vdash C \qquad \gamma \notin \mathbf{FC}(C)}{\Gamma \vdash B \to C}$$

Take $v \succcurlyeq w$ and $d \succeq_v \rho(\mathrm{pos}(\Gamma))$, and suppose $v,\, d \Vdash^{\rho} B$. By Corollary C.2 we have $v \Vdash^{\rho} \Gamma$, so letting $\rho' = \rho \cdot [\gamma \mapsto d]$ we obtain $v \Vdash^{\rho'} \Gamma, x :^{\gamma} B$. By the IH, $v,\, d \Vdash^{\rho'} C$ holds, and so does $v,\, d \Vdash^{\rho} C$ as $\gamma \notin \mathbf{FC}(C)$, which implies $w,\, \rho(\mathrm{pos}(\Gamma)) \Vdash^{\rho} B \to C$.

▷ Case (→-E). Assume

$$\frac{\Gamma \vdash B \to C \qquad \Gamma \vdash B}{\Gamma \vdash C}$$

By the IH, we have $w,\, \rho(\mathrm{pos}(\Gamma)) \Vdash^{\rho} B \to C$ and $w,\, \rho(\mathrm{pos}(\Gamma)) \Vdash^{\rho} B$. Since $w \preccurlyeq w$ and $\rho(\mathrm{pos}(\Gamma)) \preceq_w \rho(\mathrm{pos}(\Gamma))$, we obtain $w,\, \rho(\mathrm{pos}(\Gamma)) \Vdash^{\rho} C$.

▷ Case (□-I). Assume

$$\frac{\Gamma, \blacktriangleright^{\gamma_2 :\succeq \gamma_1} \vdash B \qquad \gamma_2 \notin \mathbf{FC}(B)}{\Gamma \vdash \Box^{\succeq \gamma_1} B}$$

Take $v \succcurlyeq w$ and $d \sqsupseteq_v \rho(\mathrm{pos}(\Gamma))$. By Corollary C.2 we have $v \Vdash^{\rho} \Gamma$, so letting $\rho' = \rho \cdot [\gamma_2 \mapsto d]$ we obtain $v \Vdash^{\rho'} \Gamma, \blacktriangleright^{\gamma_2 :\succeq \gamma_1}$. By the IH, $v,\, d \Vdash^{\rho'} B$ holds, and so does $v,\, d \Vdash^{\rho} B$ as $\gamma_2 \notin \mathbf{FC}(B)$, which implies $w,\, \rho(\mathrm{pos}(\Gamma)) \Vdash^{\rho} \Box^{\succeq \gamma_1} B$.

▷ Case (□-E). Assume

$$\frac{\Gamma, \blacktriangleleft^{\gamma} \vdash \Box^{\succeq \gamma_1} B \qquad \Gamma \vdash \gamma_1 \preceq \mathrm{pos}(\Gamma)}{\Gamma \vdash B}$$

From $w \Vdash^{\rho} \Gamma$, we have $w \Vdash^{\rho} \Gamma, \blacktriangleleft^{\gamma}$, and by (2), also $\rho(\gamma_1) \preceq_w \rho(\mathrm{pos}(\Gamma))$. Since $\Gamma, \blacktriangleleft^{\gamma}$ is well-formed, there should be a subderivation of $\Gamma \vdash \gamma \sqsubseteq \mathrm{pos}(\Gamma)$, which gives $\rho(\gamma) \sqsubseteq_w \rho(\mathrm{pos}(\Gamma))$ by (3). Applying the IH to $\Gamma, \blacktriangleleft^{\gamma} \vdash \Box^{\succeq \gamma_1} B$, we have $w,\, \rho(\gamma) \Vdash^{\rho} \Box^{\succeq \gamma_1} B$, and from $w \preccurlyeq w$, we see $w,\, \rho(\mathrm{pos}(\Gamma)) \Vdash^{\rho} B$.

▷ Case ($\forall$-I). Assume

$$\frac{\Gamma,\ \gamma_2 :\succeq \gamma_1 \vdash B}{\Gamma \vdash \forall \gamma_2 :\succeq \gamma_1.\ B}$$

Take $v \succcurlyeq w$ and $d \succeq_v \rho(\gamma_1)$. By Corollary C.2 we have $v \Vdash^{\rho} \Gamma$, so letting $\rho' = \rho \cdot [\gamma_2 \mapsto d]$ we obtain $v \Vdash^{\rho'} \Gamma,\ \gamma_2 :\succeq \gamma_1$. By the IH, $v,\ \rho(\mathrm{pos}(\Gamma)) \Vdash^{\rho'} B$ holds, which implies $w,\ \rho(\mathrm{pos}(\Gamma)) \Vdash^{\rho} \forall \gamma_2 :\succeq \gamma_1.\ B$.

▷ Case ($\forall$-E). Assume

$$\frac{\Gamma \vdash \forall \gamma_2 :\succeq \gamma_1.\ B \qquad \Gamma \vdash \gamma_1 \preceq \gamma}{\Gamma \vdash B[\gamma_2 := \gamma]}$$

By the IH, we have $w,\ \rho(\mathrm{pos}(\Gamma)) \Vdash^{\rho} \forall \gamma_2 :\succeq \gamma_1.\ B$, and from (2), $\rho(\gamma_1) \preceq_w \rho(\gamma)$ holds. Since $w \preccurlyeq w$, we obtain $w,\ \rho(\mathrm{pos}(\Gamma)) \Vdash^{\rho \cdot [\gamma_2 \mapsto \rho(\gamma)]} B$, and Lemma C.3 yields $w,\ \rho(\mathrm{pos}(\Gamma)) \Vdash^{\rho} B[\gamma_2 := \gamma]$. ◀

▶ **Lemma C.4.** *The following rules are admissible:*

1. *Inversion of* $\Box$-I*:*

$$\frac{\Gamma \vdash \Box^{\succeq \gamma} A}{\Gamma,\ \blacktriangleright^{\gamma' :\succeq \gamma} \vdash A}$$

2. *General weakening for top-level subderivations:*

$$\frac{\Gamma,\ \blacktriangleleft^{!},\ \Gamma' \vdash A}{\Gamma,\ \Delta,\ \blacktriangleleft^{!},\ \Gamma' \vdash A} \qquad \frac{\Gamma,\ \blacktriangleleft^{!},\ \Gamma' \vdash \gamma_1 \trianglelefteq \gamma_2}{\Gamma,\ \Delta,\ \blacktriangleleft^{!},\ \Gamma' \vdash \gamma_1 \trianglelefteq \gamma_2}$$

where $\trianglelefteq\ \in \{\preceq, \sqsubseteq\}$.

Proof.

1. Using Lemma 3.2, we have

$$\begin{array}{c} \Gamma \vdash \Box^{\succeq \gamma} A \\ \hdashline \Gamma,\ \blacktriangleright^{\gamma' :\succeq \gamma},\ \blacktriangleleft^{\mathrm{pos}(\Gamma)} \vdash \Box^{\succeq \gamma} A \\ \hline \Gamma,\ \blacktriangleright^{\gamma' :\succeq \gamma} \vdash A \end{array}\ \Box\text{-E}$$

2. By induction on derivation. ◀

▶ **Lemma 5.4** (Truth Lemma (On page 10)). $\Gamma,\ \gamma \Vdash^{\mathsf{c}} A \iff \Gamma,\ \blacktriangleleft^{!} \vdash \Box^{\succeq \gamma} A$.

Proof. By induction on the size of A, using Lemma C.4.

▷ Case ($A \equiv p$). By definition.

▷ Case ($A \equiv B \to C$).

($\Longleftarrow$) Suppose $\Gamma, \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}(B \to C)$. Take $\Delta \succsim^{\mathsf{c}} \Gamma$ and $\delta \succeq^{\mathsf{c}}_{\Delta} \gamma$ satisfying $\Delta, \delta \Vdash^{\rho^{\mathsf{c}}_{\Gamma}} B$. Since $\rho^{\mathsf{c}}_{\Delta} \upharpoonright_{\mathbf{Dom}_{\mathrm{C}}(\Gamma)} = \rho^{\mathsf{c}}_{\Gamma}$, we have $\Delta, \delta \Vdash^{\mathsf{c}} B$, and the IH yields $\Delta, \blacktriangleleft^{!} \vdash \Box^{\succeq\delta} B$. Then we can derive

$$\dfrac{\dfrac{\begin{array}{c}\Gamma, \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}(B \to C)\\ \hdashline \Delta, \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}(B \to C)\\ \hdashline \Delta, \blacktriangleleft^{!}, \blacktriangleright^{\delta':\succeq\delta} \vdash B \to C\end{array} \qquad \begin{array}{c}\Delta, \blacktriangleleft^{!} \vdash \Box^{\succeq\delta} B\\ \hdashline \Delta, \blacktriangleleft^{!}, \blacktriangleright^{\delta':\succeq\delta} \vdash B\end{array}}{\Delta, \blacktriangleleft^{!}, \blacktriangleright^{\delta':\succeq\delta} \vdash C}\,\to\text{-E}}{\Delta, \blacktriangleleft^{!} \vdash \Box^{\succeq\delta} C}\,\Box\text{-I}$$

By the IH, $\Delta, \delta \Vdash^{\mathsf{c}} C$ holds, and so does $\Delta, \delta \Vdash^{\rho^{\mathsf{c}}_{\Gamma}} C$, which implies $\Gamma, \gamma \Vdash^{\mathsf{c}} B \to C$.

($\Longrightarrow$) We argue by contrapositive. Suppose $\Gamma, \blacktriangleleft^{!} \nvdash \Box^{\succeq\gamma}(B \to C)$. Let $\Delta \equiv \Gamma, \blacktriangleleft^{!}, \blacktriangleright^{\gamma':\succeq\gamma}$, $x :^{\delta} B$. Then we must have $\Delta, \blacktriangleleft^{!} \nvdash \Box^{\succeq\delta} C$; otherwise we could derive

$$\dfrac{\dfrac{\dfrac{\Delta, \blacktriangleleft^{!} \vdash \Box^{\succeq\delta} C}{\Gamma, \blacktriangleleft^{!}, \blacktriangleright^{\gamma':\succeq\gamma}, x :^{\delta} B \vdash C}\,\Box\text{-E}}{\Gamma, \blacktriangleleft^{!}, \blacktriangleright^{\gamma':\succeq\gamma} \vdash B \to C}\,\to\text{-I}}{\Gamma, \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}(B \to C)}\,\Box\text{-I}$$

a contradiction. By the IH, we have $\Delta, \delta \nVdash^{\mathsf{c}} C$, and also $\Delta, \delta \Vdash^{\rho^{\mathsf{c}}_{\Gamma}} C$ as $\mathbf{FC}(C) \subseteq \mathbf{Dom}_{\mathrm{C}}(\Gamma)$. Applying a similar argument to

$$\dfrac{\dfrac{}{\Delta, \blacktriangleleft^{!}, \blacktriangleright^{\delta':\succeq\delta} \vdash B}\,\text{Var}}{\Delta, \blacktriangleleft^{!} \vdash \Box^{\succeq\delta} B}\,\Box\text{-I}$$

yields $\Delta, \delta \Vdash^{\rho^{\mathsf{c}}_{\Gamma}} B$. Since $\Delta \succsim^{\mathsf{c}} \Gamma$ and $\delta \succeq^{\mathsf{c}}_{\Delta} \gamma$, we see $\Gamma, \gamma \nVdash^{\mathsf{c}} B \to C$.

▷ Case ($A \equiv \Box^{\succeq\gamma_1}B$).

($\Longleftarrow$) Suppose $\Gamma,\ \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}\Box^{\succeq\gamma_1}B$. Take $\Delta \succcurlyeq^{\mathsf{c}} \Gamma$ and $\delta \sqsupseteq^{\mathsf{c}}_{\Delta} \gamma$ satisfying $\delta \succeq^{\mathsf{c}}_{\Delta} \gamma_1$. Then we can derive

$$
\begin{array}{cl}
\Gamma,\ \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}\Box^{\succeq\gamma_1}B & \\
\hdashline
\Delta,\ \blacktriangleleft^{!},\ \blacktriangleright^{\delta':\succeq\delta},\ \blacktriangleleft^{\gamma},\ \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}\Box^{\succeq\gamma_1}B & \\
\cline{1-1}
\Delta,\ \blacktriangleleft^{!},\ \blacktriangleright^{\delta':\succeq\delta},\ \blacktriangleleft^{\gamma} \vdash \Box^{\succeq\gamma_1}B & \Box\text{-E} \\
\cline{1-1}
\Delta,\ \blacktriangleleft^{!},\ \blacktriangleright^{\delta':\succeq\delta} \vdash B & \Box\text{-E} \\
\cline{1-1}
\Delta,\ \blacktriangleleft^{!} \vdash \Box^{\succeq\delta}B & \Box\text{-I}
\end{array}
$$

By the IH, $\Delta,\ \delta \Vdash^{\mathsf{c}} B$ holds, and so does $\Delta,\ \delta \Vdash^{\rho^{\mathsf{c}}_{\Gamma}} B$, which implies $\Gamma,\ \gamma \Vdash^{\mathsf{c}} \Box^{\succeq\gamma_1}B$.

($\Longrightarrow$) We argue by contrapositive. Suppose $\Gamma,\ \blacktriangleleft^{!} \nvdash \Box^{\succeq\gamma}\Box^{\succeq\gamma_1}B$. Let $\Delta \equiv \Gamma,\ \blacktriangleleft^{!},\ \blacktriangleright^{\gamma':\succeq\gamma},\ \blacktriangleright^{\delta:\succeq\gamma_1}$. Then we must have $\Delta,\ \blacktriangleleft^{!} \nvdash \Box^{\succeq\delta}B$; otherwise we could derive

$$
\begin{array}{cl}
\Delta,\ \blacktriangleleft^{!} \vdash \Box^{\succeq\delta}B & \\
\cline{1-1}
\Gamma,\ \blacktriangleleft^{!},\ \blacktriangleright^{\gamma':\succeq\gamma},\ \blacktriangleright^{\delta:\succeq\gamma_1} \vdash B & \Box\text{-E} \\
\cline{1-1}
\Gamma,\ \blacktriangleleft^{!},\ \blacktriangleright^{\gamma':\succeq\gamma} \vdash \Box^{\succeq\gamma_1}B & \Box\text{-I} \\
\cline{1-1}
\Gamma,\ \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}\Box^{\succeq\gamma_1}B & \Box\text{-I}
\end{array}
$$

a contradiction. By the IH, we have $\Delta,\ \delta \nVdash^{\mathsf{c}} B$ and hence $\Delta,\ \delta \Vdash^{\rho^{\mathsf{c}}_{\Gamma}} B$. As $\Delta \succcurlyeq^{\mathsf{c}} \Gamma$ and $\delta \sqsupseteq^{\mathsf{c}}_{\Delta} \gamma$ with $\delta \succeq^{\mathsf{c}}_{\Delta} \gamma_1$, we see $\Gamma,\ \gamma \nVdash^{\mathsf{c}} \Box^{\succeq\gamma_1}B$.

▷ Case ($A \equiv \forall\gamma_2 :\succeq \gamma_1.\ B$).

($\Longleftarrow$) Suppose $\Gamma,\ \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}(\forall\gamma_2 :\succeq \gamma_1.\ B)$. Take $\Delta \succcurlyeq^{\mathsf{c}} \Gamma$ and $\delta \succeq^{\mathsf{c}}_{\Delta} \gamma_1$. Then we can derive

$$
\begin{array}{cl}
\Gamma,\ \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}(\forall\gamma_2 :\succeq \gamma_1.\ B) & \\
\hdashline
\Delta,\ \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}(\forall\gamma_2 :\succeq \gamma_1.\ B) & \\
\hdashline
\Delta,\ \blacktriangleleft^{!},\ \blacktriangleright^{\gamma':\succeq\gamma} \vdash \forall\gamma_2 :\succeq \gamma_1.\ B & \\
\cline{1-1}
\Delta,\ \blacktriangleleft^{!},\ \blacktriangleright^{\gamma':\succeq\gamma} \vdash B[\gamma_2 := \delta] & \forall\text{-E} \\
\cline{1-1}
\Delta,\ \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}B[\gamma_2 := \delta] & \Box\text{-I}
\end{array}
$$

By the IH, $\Delta,\ \gamma \Vdash^{\mathsf{c}} B[\gamma_2 := \delta]$ holds, and so does $\Delta,\ \gamma \Vdash^{\rho^{\mathsf{c}}_{\Gamma}\cdot[\gamma_2\mapsto\delta]} B$, which yields $\Gamma,\ \gamma \Vdash^{\mathsf{c}} \forall\gamma_2 :\succeq \gamma_1.\ B$.

($\Longrightarrow$) We argue by contrapositive. Suppose $\Gamma,\ \blacktriangleleft^{!} \nvdash \Box^{\succeq\gamma}(\forall\gamma_2 :\succeq \gamma_1.\ B)$. Let $\Delta \equiv \Gamma,\ \blacktriangleleft^{!},\ \blacktriangleright^{\gamma':\succeq\gamma},\ \gamma_2 :\succeq \gamma_1$. Then we must have $\Delta,\ \blacktriangleleft^{!} \nvdash \forall\gamma_2 :\succeq \gamma_1.\ B$; otherwise we could derive

$$
\begin{array}{cl}
\Delta,\ \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}B & \\
\cline{1-1}
\Gamma,\ \blacktriangleleft^{!},\ \blacktriangleright^{\gamma':\succeq\gamma},\ \gamma_2 :\succeq \gamma_1 \vdash B & \Box\text{-E} \\
\cline{1-1}
\Gamma,\ \blacktriangleleft^{!},\ \blacktriangleright^{\gamma':\succeq\gamma} \vdash \forall\gamma_2 :\succeq \gamma_1.\ B & \forall\text{-I} \\
\cline{1-1}
\Gamma,\ \blacktriangleleft^{!} \vdash \Box^{\succeq\gamma}(\forall\gamma_2 :\succeq \gamma_1.\ B) & \Box\text{-I}
\end{array}
$$

a contradiction. By the IH, we have $\Delta,\ \gamma \nVdash^{\mathsf{c}} B$, and hence $\Delta,\ \gamma \nVdash^{\rho^{\mathsf{c}}_{\Gamma}\cdot[\gamma_2\mapsto\gamma_2]} B$. Since $\Delta \succcurlyeq^{\mathsf{c}} \Gamma$, we see $\Gamma,\ \gamma \Vdash^{\mathsf{c}} \forall\gamma_2 :\succeq \gamma_1.\ B$. ◀

▶ **Theorem 5.5** (Kripke Completeness (On page 10)).

1. *If* $\Gamma \Vdash A$, *then* $\Gamma \vdash A$.
2. *If* $\Gamma \Vdash \gamma_1 \preceq \gamma_2$, *then* $\Gamma \vdash \gamma_1 \preceq \gamma_2$.
3. *If* $\Gamma \Vdash \gamma_1 \sqsubseteq \gamma_2$, *then* $\Gamma \vdash \gamma_1 \sqsubseteq \gamma_2$.

Proof. By the contrapositive, where we can take the canonical model as countermodel by Lemma 5.4. ◀

D Full Definitions and Proofs for Section 6

▶ **Definition D.1.** $\mathbf{FC}(M)$ *represent a set of free classifiers in* M*.*

$$\begin{aligned}
\mathbf{FC}(x) &= \emptyset \\
\mathbf{FC}(\lambda x :^{\gamma} A.\, M) &= \mathbf{FC}(M) \cup \{\gamma\} \\
\mathbf{FC}(M_1 M_2) &= \mathbf{FC}(M_1) \cup \mathbf{FC}(M_2) \\
\mathbf{FC}(\mathbf{quo}\{^{\gamma_1 :\succeq \gamma_2} M\}) &= (\mathbf{FC}(M) \cup \{\gamma_2\}) - \{\gamma_1\} \\
\mathbf{FC}(\mathbf{unq}\{^{\gamma} M\}) &= \mathbf{FC}(M) \cup \{\gamma\} \\
\mathbf{FC}(\lambda \gamma_1 :\succeq \gamma_2.\ M) &= (\mathbf{FC}(M) - \{\gamma_1\}) \cup \{\gamma_2\} \\
\mathbf{FC}(M \gamma) &= \mathbf{FC}(M) \cup \{\gamma\}
\end{aligned}$$

▶ **Definition D.2.** $\mathbf{FV}(M)$ *represents a set of free variables in* M*.*

$$\begin{aligned}
\mathbf{FV}(x) &= \{x\} \\
\mathbf{FV}(\lambda x :^{\gamma} A.\, M) &= \mathbf{FV}(M) - \{x\} \\
\mathbf{FV}(M_1 M_2) &= \mathbf{FV}(M_1) \cup \mathbf{FV}(M_2) \\
\mathbf{FV}(\mathbf{quo}\{^{\gamma_1 :\succeq \gamma_2} M\}) &= \mathbf{FV}(M) \\
\mathbf{FV}(\mathbf{unq}\{^{\gamma} M\}) &= \mathbf{FV}(M) \\
\mathbf{FV}(\lambda \gamma_1 :\succeq \gamma_2.\ M) &= \mathbf{FV}(M) \\
\mathbf{FV}(M \gamma) &= \mathbf{FV}(M)
\end{aligned}$$

▶ **Definition D.3.** *A classifier substitution* $(-)[\gamma_1 \coloneqq \gamma_2]$ *is a meta operation on terms and contexts, which replaces free occurrences of* γ_1 *with* γ_2*.*

$$\begin{aligned}
x[\gamma_1 \coloneqq \gamma_2] &= x \\
(\lambda x :^{\gamma_1} A.\, M)[\gamma_2 \coloneqq \gamma_3] &= \lambda x :^{\gamma_1} A[\gamma_2 \coloneqq \gamma_3].\, (M[\gamma_2 \coloneqq \gamma_3]) \\
&\qquad \textit{where } \gamma_1 \notin \{\gamma_2, \gamma_3\} \\
(MN)[\gamma_1 \coloneqq \gamma_2] &= (M[\gamma_1 \coloneqq \gamma_2])(N[\gamma_1 \coloneqq \gamma_2]) \\
\mathbf{quo}\{^{\gamma_1 :\succeq \gamma_2} M\}[\gamma_3 \coloneqq \gamma_4] &= \mathbf{quo}\{^{\gamma_1 :\succeq \gamma_2[\gamma_3 \coloneqq \gamma_4]} M[\gamma_3 \coloneqq \gamma_4]\} \\
&\qquad \textit{where } \gamma_1 \notin \{\gamma_3, \gamma_4\} \\
\mathbf{unq}\{^{\gamma_1} M\}[\gamma_2 \coloneqq \gamma_3] &= \mathbf{unq}\{^{\gamma_1[\gamma_2 \coloneqq \gamma_3]} M[\gamma_1 \coloneqq \gamma_2]\} \\
(\lambda \gamma_1 :\succeq \gamma_2.\ M)[\gamma_3 \coloneqq \gamma_4] &= \lambda \gamma_1 :\succeq \gamma_2[\gamma_3 \coloneqq \gamma_4].\ (M[\gamma_3 \coloneqq \gamma_4]) \\
&\qquad \textit{where } \gamma_1 \notin \{\gamma_3, \gamma_4\} \\
(M \gamma_1)[\gamma_2 \coloneqq \gamma_3] &= (M[\gamma_2 \coloneqq \gamma_3])\gamma_1[\gamma_2 \coloneqq \gamma_3]
\end{aligned}$$

$$\varepsilon[\gamma_1 \coloneqq \gamma_2] = \varepsilon$$

$$
\begin{aligned}
(x :^{\gamma_1} A, \Gamma)[\gamma_2 \coloneqq \gamma_3] &= x :^{\gamma_1} A[\gamma_2 \coloneqq \gamma_3], \Gamma[\gamma_2 \coloneqq \gamma_3] \\
&\quad \textit{where } \gamma_1 \notin \{\gamma_2, \gamma_3\} \\
(\blacktriangleright^{\gamma_1 :\succeq \gamma_2}, \Gamma)[\gamma_3 \coloneqq \gamma_4] &= \blacktriangleright^{\gamma_1 :\succeq \gamma_2[\gamma_3 \coloneqq \gamma_4]}, \Gamma[\gamma_3 \coloneqq \gamma_4] \\
&\quad \textit{where } \gamma_1 \notin \{\gamma_3, \gamma_4\} \\
(\blacktriangleleft^{\gamma_1}, \Gamma)[\gamma_2 \coloneqq \gamma_3] &= \blacktriangleleft^{\gamma_1[\gamma_2 \coloneqq \gamma_3]}, \Gamma[\gamma_2 \coloneqq \gamma_3] \\
(\gamma_1 :\succeq \gamma_2, \Gamma)[\gamma_3 \coloneqq \gamma_4] &= \gamma_1 :\succeq \gamma_2[\gamma_3 \coloneqq \gamma_4], \Gamma[\gamma_3 \coloneqq \gamma_4] \\
&\quad \textit{where } \gamma_1 \notin \{\gamma_3, \gamma_4\}
\end{aligned}
$$

▶ **Definition D.4.** *A variable substitution* $(-)[\gamma_1 \coloneqq \gamma_2, x \coloneqq M]$ *is a meta operation on terms that replaces free occurrences of* γ_1 *and* x *with* γ_2 *and* M*, respectively. Its definition is given in Figure 10.*

$$
\begin{aligned}
x[\gamma_1 \coloneqq \gamma_2, y \coloneqq M] &= \begin{cases} M & \text{where } x = y \\ x & \text{otherwise} \end{cases} \\
(\lambda x :^{\gamma_1} A.\, M)[\gamma_2 \coloneqq \gamma_3, y \coloneqq M] &= \lambda x :^{\gamma_1} A[\gamma_2 \coloneqq \gamma_3].\, (M[\gamma_2 \coloneqq \gamma_3, y \coloneqq M]) \\
&\quad \text{where } \gamma_1 \notin \{\gamma_2, \gamma_3\} \text{ and } x \neq y \\
(M_1 M_2)[\gamma_1 \coloneqq \gamma_2, x \coloneqq N] &= M_1[\gamma_1 \coloneqq \gamma_2, x \coloneqq N] M_2[\gamma_1 \coloneqq \gamma_2, y \coloneqq N] \\
\mathbf{quo}\{^{\gamma_1 :\succeq \gamma_2} M\}[\gamma_3 \coloneqq \gamma_4, x \coloneqq N] &= \mathbf{quo}\{^{\gamma_1 :\succeq \gamma_2[\gamma_3 \coloneqq \gamma_4]} M[\gamma_3 \coloneqq \gamma_4, x \coloneqq N]\} \\
&\quad \text{where } \gamma_1 \notin \{\gamma_3, \gamma_4\} \\
\mathbf{unq}\{^{\gamma_1} M\}[\gamma_2 \coloneqq \gamma_3, x \coloneqq N] &= \mathbf{unq}\{^{\gamma_1[\gamma_2 \coloneqq \gamma_3]} M[\gamma_2 \coloneqq \gamma_3, x \coloneqq N]\} \\
(\lambda \gamma_1 :\succeq \gamma_2.\, M)[\gamma_3 \coloneqq \gamma_4, x \coloneqq N] &= \lambda \gamma_1 :\succeq \gamma_2[\gamma_3 \coloneqq \gamma_4].\, (M[\gamma_3 \coloneqq \gamma_4, x \coloneqq N]) \\
&\quad \text{where } \gamma_1 \notin \{\gamma_3, \gamma_4\} \\
(M \gamma_1)[\gamma_2 \coloneqq \gamma_3, x \coloneqq N] &= M[\gamma_2 \coloneqq \gamma_3, x \coloneqq N] \gamma_1[\gamma_2 \coloneqq \gamma_3]
\end{aligned}
$$

Figure 10 Definition of Variable Substitution

Lemmas 6.1–6.3 are proved as parts of the following lemmas.

▶ **Lemma D.5** (Variable Substitution (Full Version)). *Suppose* $\Delta_1 = \Gamma_1^{\gamma_1}, x :^{\gamma_2} A, \Gamma_2$*, and* $\Delta_2 = \Gamma_1, \Gamma_2[\gamma_2 \coloneqq \gamma_1]$*. Then, the following statements hold.*

1. $\vdash \Delta_1 : \mathbf{ctx} \implies \vdash \Delta_2 : \mathbf{ctx}$.
2. $\Delta_1 \vdash A : \mathbf{type} \implies \Delta_2 \vdash A[\gamma_2 \coloneqq \gamma_1] : \mathbf{type}$.
3. $\Delta_1 \vdash \delta_1 \preceq \delta_2 \implies \Delta_2 \vdash \delta_1[\gamma_2 \coloneqq \gamma_1] \preceq \delta_2[\gamma_2 \coloneqq \gamma_1]$.
4. $\Delta_1 \vdash \delta_1 \sqsubseteq \delta_2 \implies \Delta_2 \vdash \delta_1[\gamma_2 \coloneqq \gamma_1] \sqsubseteq \delta_2[\gamma_2 \coloneqq \gamma_1]$.
5. $\Delta_1 \vdash M_1 : B$ *and* $\Gamma_1 \vdash M_2 : A \implies \Delta_2 \vdash M_1[\gamma_2 \coloneqq \gamma_1, x \coloneqq M_2] : B[\gamma_2 \coloneqq \gamma_1]$.

Proof. By mutual induction on the derivation of the first judgment for each statements. To prove the case of typing judgment, we use Lemma 3.2 for the base case where M_1 is variable. ◀

▶ **Lemma D.6** (Rebasing (Full Version)). *Suppose* $\Delta_1 = (\Gamma_1^{\gamma_1}, \blacktriangleleft^{\gamma_2}, \blacktriangleright^{\gamma_3 :\succeq \gamma_4}, \Gamma_2)$*, and* $\Delta_2 = \Gamma_1, \Gamma_2[\gamma_3 \coloneqq \gamma_1]$*. Supposing* $\Gamma_1 \vdash \gamma_4 \preceq \gamma_1$*, the following statements hold,*

1. $\vdash \Delta_1 : \mathbf{ctx} \implies \vdash \Delta_2 : \mathbf{ctx}$.
2. $\Delta_1 \vdash A : \mathbf{type} \implies \Delta_2 \vdash A[\gamma_3 \coloneqq \gamma_1] : \mathbf{type}$.

3. $\Delta_1 \vdash \delta_1 \preceq \delta_2 \implies \Delta_2 \vdash \delta_1[\gamma_3 \coloneqq \gamma_1] \preceq \delta_2[\gamma_3 \coloneqq \gamma_1]$.
4. $\Delta_1 \vdash \delta_1 \sqsubseteq \delta_2 \implies \Delta_2 \vdash \delta_1[\gamma_3 \coloneqq \gamma_1] \sqsubseteq \delta_2[\gamma_3 \coloneqq \gamma_1]$.
5. $\Delta_1 \vdash M_1 : A \implies \Delta_2 \vdash M[\gamma_3 \coloneqq \gamma_1] : A[\gamma_3 \coloneqq \gamma_1]$.

Proof. By mutual induction on the first derivation of each statements. ◀

▶ **Lemma D.7** (Classifier Substitution (Full Version)). *Suppose* $\Delta_1 = \Gamma_1, \gamma_1 :\succeq \gamma_2, \Gamma_2$ *and* $\Delta_2 = \Gamma_1, \Gamma_2[\gamma_1 \coloneqq \gamma_3]$. *Given* $\Gamma_1 \vdash \gamma_2 \preceq \gamma_3$, *then the following statements hold.*

1. $\Delta_1 \vdash A : \mathbf{type} \implies \Delta_2 \vdash A[\gamma_1 \coloneqq \gamma_3] : \mathbf{type}$.
2. $\vdash \Delta_1 : \mathbf{ctx} \implies \vdash \Delta_2 : \mathbf{ctx}$.
3. $\Delta_1 \vdash \delta_1 \preceq \delta_2 \implies \Delta_2 \vdash \delta_1[\gamma_1 \coloneqq \gamma_3] \preceq \delta_2[\gamma_1 \coloneqq \gamma_3]$.
4. $\Delta_1 \vdash \delta_1 \sqsubseteq \delta_2 \implies \Delta_2 \vdash \delta_1[\gamma_1 \coloneqq \gamma_3] \sqsubseteq \delta_2[\gamma_1 \coloneqq \gamma_3]$.
5. $\Delta_1 \vdash M : A \implies \Delta_2 \vdash M[\gamma_1 \coloneqq \gamma_3] : A[\gamma_1 \coloneqq \gamma_3]$.

Proof. By mutual induction on the first derivation of each statements. ◀

▶ **Lemma 6.4** (Local Soundness Patterns (On page 11)).

1. $\Gamma^{\gamma_1} \vdash (\lambda x :^{\gamma_2} A.\, M)N : B \implies \Gamma \vdash M[\gamma_2 \coloneqq \gamma_1, x \coloneqq N] : B$.
2. $\Gamma^{\gamma_1} \vdash \mathbf{unq}\{^{\gamma_2}\mathbf{quo}\{^{\gamma_3 :\succeq \gamma_4} M\}\} : A \implies \Gamma \vdash M[\gamma_3 \coloneqq \gamma_1] : A$.
3. $\Gamma \vdash (\lambda \gamma_1 :\succeq \gamma_2.\ M)\gamma_3 : A \implies \Gamma \vdash M[\gamma_1 \coloneqq \gamma_3] : A$.

Proof. Easy to prove with Lemma 6.1, Lemma 6.2 and Lemma 6.3. ◀

▶ **Lemma 6.5** (Local Completeness Patterns (On page 11)). *(δ is taken freshly)*

1. $\Gamma \vdash M : A \to B \implies \Gamma \vdash \lambda x :^{\delta} A.\, (Mx) : A \to B$.
2. $\Gamma^{\gamma_1} \vdash M : \Box^{\succeq \gamma_2} A \implies \Gamma \vdash \mathbf{quo}\{^{\delta :\succeq \gamma_2}\mathbf{unq}\{^{\gamma_1} M\}\} : \Box^{\succeq \gamma_2} A$.
3. $\Gamma \vdash M : \forall \gamma_1 :\succeq \gamma_2.\ A \implies \Gamma \vdash \lambda \delta :\succeq \gamma_2.\ (M\delta) : \forall \gamma_1 :\succeq \gamma_2.\ A$.

Proof. Easy to prove with Lemma 3.2. ◀

▶ **Definition D.8** (Full rules for Definition 6.6 (β-reduction)). *The full definition of derivation rules for* $M_1 \Rightarrow_\beta^\gamma M_2$ *are follows.*

Axioms

$$(\lambda x :^{\gamma_2} A.\, M)N@\gamma_1 \Rightarrow_\beta M[\gamma_2 \coloneqq \gamma_1, x \coloneqq N]$$

$$\mathbf{unq}\{^{\gamma_2}\mathbf{quo}\{^{\gamma_3 :\succeq \gamma_4} M\}\}@\gamma_1 \Rightarrow_\beta M[\gamma_3 \coloneqq \gamma_1]$$

$$(\lambda \gamma_2 :\succeq \gamma_3.\ M)\gamma_4@\gamma_1 \Rightarrow_\beta M[\gamma_2 \coloneqq \gamma_4]$$

Compatibility Rules

$$\frac{M_1 \Rightarrow_\beta^{\gamma_1} M_2}{\lambda x :^{\gamma_1} A.\, M_1 \Rightarrow_\beta^{\gamma_2} \lambda x :^{\gamma_1} A.\, M_2} \qquad \frac{M_1 \Rightarrow_\beta^{\gamma_1} M_2}{M_1 N \Rightarrow_\beta^{\gamma_1} M_2 N} \qquad \frac{M_1 \Rightarrow_\beta^{\gamma_1} M_2}{N M_1 \Rightarrow_\beta^{\gamma_1} N M_2}$$

$$\frac{M_1 \Rightarrow_\beta^{\gamma_1} M_2}{\mathbf{quo}\{^{\gamma_1 :\succeq \gamma_2} M_1\} \Rightarrow_\beta^{\gamma_3} \mathbf{quo}\{^{\gamma_1 :\succeq \gamma_2} M_2\}} \qquad \frac{M_1 \Rightarrow_\beta^{\gamma_1} M_2}{\mathbf{unq}\{^{\gamma_1} M_1\} \Rightarrow_\beta^{\gamma_2} \mathbf{unq}\{^{\gamma_1} M_2\}}$$

$$\frac{M_1 \Rightarrow_\beta^{\gamma_1} M_2}{\lambda \gamma_2 :\succeq \gamma_3.\ M_1 \Rightarrow_\beta^{\gamma_1} \lambda \gamma_2 :\succeq \gamma_3.\ M_2} \qquad \frac{M_1 \Rightarrow_\beta^{\gamma_1} M_2}{M_1 \gamma_2 \Rightarrow_\beta^{\gamma_1} M_2 \gamma_2}$$

▶ **Theorem 6.7** (Subject Reduction (On page 12)). *If* $\Gamma^{\gamma} \vdash M_1 : A$ *and* $M_1 \Rightarrow_\beta^\gamma M_2$, *then* $\Gamma \vdash M_2 : A$.

Proof. By induction on the derivation of $M_1 \Rightarrow_\beta^\gamma M_2$. For base cases, we apply Lemma 6.4. ◀

$\mathcal{E}$-id
$$\frac{\Gamma \vdash x : A}{\Gamma \vdash x \in \mathcal{E}\llbracket A \rrbracket}$$

$\mathcal{E}$-blur
$$\frac{\forall \Delta \succcurlyeq \Gamma.\ \big(\Delta \vdash K \in \mathcal{K}\llbracket A \rrbracket \implies K[M] \in \mathrm{SN}\big)}{\Gamma \vdash M \in \mathcal{E}\llbracket A \rrbracket}$$

$\mathcal{K}$-id
$$\frac{}{\Gamma \vdash [-] \in \mathcal{K}\llbracket A \rrbracket}$$

$\mathcal{K}$-$\rightarrow$
$$\frac{\Gamma \vdash N \in \mathcal{E}\llbracket A \rrbracket \qquad \Gamma \vdash K \in \mathcal{K}\llbracket B \rrbracket}{\Gamma \vdash K[[-]N] \in \mathcal{K}\llbracket A \rightarrow B \rrbracket}$$

$\mathcal{K}$-$\Box$
$$\frac{\Gamma \vdash \gamma \sqsubseteq \mathrm{pos}(\Gamma) \qquad \Gamma \vdash \gamma_1 \preceq \mathrm{pos}(\Gamma) \qquad \Gamma \vdash K \in \mathcal{K}\llbracket A \rrbracket}{\Gamma,\ \blacktriangleleft^{\gamma} \vdash K[\mathbf{unq}\{^{\gamma}\,[-]\}] \in \mathcal{K}\llbracket \Box^{\succeq \gamma_1} A \rrbracket}$$

$\mathcal{K}$-$\forall$
$$\frac{\Gamma \vdash \gamma_1 \preceq \gamma \qquad \Gamma \vdash K \in \mathcal{K}\llbracket A[\gamma_2 \coloneqq \gamma] \rrbracket}{\Gamma \vdash K[[-]\gamma] \in \mathcal{K}\llbracket \forall \gamma_2 :\succeq \gamma_1.\ A \rrbracket}$$

$\mathcal{C}$-id
$$\frac{}{\Gamma \vdash \emptyset \in \mathcal{C}\llbracket \Gamma \rrbracket}$$

$\mathcal{C}$-weak
$$\frac{\Delta \vdash \sigma \in \mathcal{C}\llbracket \Gamma \rrbracket \qquad \Delta \preccurlyeq \Delta'}{\Delta' \vdash \sigma \in \mathcal{C}\llbracket \Gamma \rrbracket}$$

$\mathcal{C}$-$\rightarrow$
$$\frac{\Delta \vdash \sigma \in \mathcal{C}\llbracket \Gamma \rrbracket \qquad \Delta \vdash M \in \mathcal{E}\llbracket A\,\sigma \rrbracket}{\Delta \vdash \sigma \cdot [\gamma \coloneqq \mathrm{pos}(\Delta),\ x \coloneqq M] \in \mathcal{C}\llbracket \Gamma,\ x :^{\gamma} A \rrbracket}$$

$\mathcal{C}$-$\Box$
$$\frac{\Delta,\ \blacktriangleleft^{\gamma} \vdash \sigma \in \mathcal{C}\llbracket \Gamma \rrbracket \qquad \Delta \vdash \gamma_1\,\sigma \preceq \mathrm{pos}(\Delta)}{\Delta \vdash \sigma \cdot [\gamma_2 \coloneqq \mathrm{pos}(\Delta)] \in \mathcal{C}\llbracket \Gamma,\ \blacktriangleright^{\gamma_2 :\succeq \gamma_1} \rrbracket}$$

$\mathcal{C}$-$\forall$
$$\frac{\Delta \vdash \sigma \in \mathcal{C}\llbracket \Gamma \rrbracket \qquad \Delta \vdash \gamma_1\,\sigma \preceq \gamma}{\Delta \vdash \sigma \cdot [\gamma_2 \coloneqq \gamma] \in \mathcal{C}\llbracket \Gamma,\ \gamma_2 :\succeq \gamma_1 \rrbracket}$$

$\mathcal{C}$-$\blacktriangleleft$
$$\frac{\Delta \vdash \sigma \in \mathcal{C}\llbracket \Gamma \rrbracket}{\Delta,\ \blacktriangleleft^{\gamma\,\sigma} \vdash \sigma \in \mathcal{C}\llbracket \Gamma,\ \blacktriangleleft^{\gamma} \rrbracket}$$

$$\Gamma \preccurlyeq \Delta \iff \begin{cases} \blacksquare\ \Gamma \vdash \gamma_1 \trianglelefteq \gamma_2 \implies \Delta \vdash \gamma_1 \trianglelefteq \gamma_2; \\ \blacksquare\ x :^{\gamma} A \in \Gamma \implies x :^{\gamma} A \in \Delta; \text{ and} \\ \blacksquare\ \Delta \vdash \mathrm{pos}(\Gamma) \preceq \mathrm{pos}(\Delta) \end{cases}$$

Figure 11 Reducibility. The predicates $\mathcal{E}\llbracket A \rrbracket$, $\mathcal{K}\llbracket A \rrbracket$, and $\mathcal{C}\llbracket \Gamma \rrbracket$ are for terms, for continuations, and for substitutions (contexts), respectively.

D.1 Metatheory

To define reducibility, we adopt the idea of using continuations from Lindley [16]. A *continuation* K is a term context defined by the following grammar:

$$K ::= [-] \mid K[[-]N] \mid K[\mathbf{unq}\{^{\gamma}\,[-]\}] \mid K[[-]\gamma].$$

Figure 11 defines the predicates $\mathcal{E}\llbracket A \rrbracket$ for terms and $\mathcal{K}\llbracket A \rrbracket$ for continuations, simultaneously inductively on A. To provide a proper computational interpretation, each predicate is indexed by a context Γ, which is particularly crucial in **BML** because: **1.** the reduction is a position-aware relation; and **2.** $\blacktriangleleft$'s cannot be removed from a context by abstraction.

The preorder $\preccurlyeq$ generalizes the canonical relation $\preccurlyeq^{\mathrm{c}}$ from Definition 5.2 while taking the position $\mathrm{pos}(\Gamma)$ into account to ensure typeability. To accommodate monotonicity, $\Gamma \vdash M \in \mathcal{E}\llbracket A \rrbracket$ verifies the SN-ability of M with all reducible continuations K under all $\preccurlyeq$-successor contexts Δ of Γ, whereas $\Gamma \vdash K \in \mathcal{K}\llbracket A \rrbracket$ is just validated locally within Γ.

▶ **Lemma D.9.** *If* $\Gamma \vdash M \in \mathcal{E}\llbracket A \rrbracket$, *then* $M \in \mathrm{SN}$.

Proof. If $\mathcal{E}$-id is applied, then obvious; otherwise follows from $\Gamma \vdash [-] \in \mathcal{K}\llbracket A \rrbracket$ by $\mathcal{K}$-id. ◀

▶ **Lemma D.10** (SN-closure)**.**

1. *If* $N \in \mathrm{SN}$ *and* $K[M[\gamma := \gamma', x := N]] \in \mathrm{SN}$*, then* $K[(\lambda x :^{\gamma} A.\ M)N] \in \mathrm{SN}$.
2. *If* $K[M[\gamma_2 := \gamma']] \in \mathrm{SN}$*, then* $K[\mathbf{unq}\{^{\gamma_3}\,\mathbf{quo}\{^{\gamma_2 :\succeq \gamma_1} M\}\}] \in \mathrm{SN}$.
3. *If* $K[M[\gamma_2 := \gamma_3]] \in \mathrm{SN}$*, then* $K[(\lambda \gamma_2 :\succeq \gamma_1.\ M)\gamma_3] \in \mathrm{SN}$.

Proof. By the contrapositive. ◀

Next, we consider the interpretation of a context. A *simultaneous substitution* σ is defined by the following grammar:

$$\sigma ::= \emptyset \mid \sigma \cdot [\gamma := \gamma', x := M] \mid \sigma \cdot [\gamma := \gamma'],$$

where the symbol $\emptyset$ denotes an *empty substitution* that does not replace any variable or classifier. Figure 11 gives the definition of a *reducible substitution* $\Delta \vdash \sigma \in \mathcal{C}[\![\Gamma]\!]$, whereby variables and classifiers declared in Γ are instantiated under Δ.

▶ **Lemma D.11** (Reducibility-closure)**.** *The following rules are admissible:*

$\mathcal{E}$-$\to$

$$\frac{\forall \Delta \curlyeqsucc \Gamma.\ \left(\begin{array}{l}\Delta \vdash N \in \mathcal{E}[\![A]\!] \implies \\ \quad \Delta \vdash M[\gamma := \mathrm{pos}(\Delta), x := N] \in \mathcal{E}[\![B]\!]\end{array}\right)}{\Gamma \vdash \lambda x :^{\gamma} A.\ M \in \mathcal{E}[\![A \to B]\!]}$$

$\mathcal{E}$-$\Box$

$$\frac{\forall (\Delta, \blacktriangleleft^{\delta}) \curlyeqsucc \Gamma.\ \left(\begin{array}{l}\Delta \vdash \delta \sqsubseteq \mathrm{pos}(\Delta) \\ \quad \text{and } \ \Delta \vdash \gamma_1 \preceq \mathrm{pos}(\Delta) \implies \\ \quad \Delta \vdash M[\gamma_2 := \mathrm{pos}(\Delta)] \in \mathcal{E}[\![A]\!]\end{array}\right)}{\Gamma \vdash \mathbf{quo}\{^{\gamma_2 :\succeq \gamma_1} M\} \in \mathcal{E}[\![\Box^{\succeq \gamma_1} A]\!]}$$

$\mathcal{E}$-$\forall$

$$\frac{\forall \Delta \curlyeqsucc \Gamma.\ \left(\begin{array}{l}\Delta \vdash \gamma_1 \preceq \gamma \implies \\ \quad \Delta \vdash M[\gamma_2 := \gamma] \in \mathcal{E}[\![A[\gamma_2 := \gamma]]\!]\end{array}\right)}{\Gamma \vdash \lambda \gamma_2 :\succeq \gamma_1.\ M \in \mathcal{E}[\![\forall \gamma_2 :\succeq \gamma_1.\ A]\!]}$$

Proof. By simply checking the condition of $\mathcal{E}$-blur.

Rule $\mathcal{E}$-$\to$. Suppose $\Delta \curlyeqsucc \Gamma$. Take $\Delta \vdash K \in \mathcal{K}[\![A \to B]\!]$ to show $K[\lambda x :^{\gamma} A.\ M] \in \mathrm{SN}$. For the last rule of the derivation of K, there are two possibilities:

▷ Case ($\mathcal{K}$-id). We may assume without loss of generality that $x \notin \mathbf{Dom}_{\mathrm{V}}(\Delta)$ and $\gamma \notin \mathbf{Dom}_{\mathrm{C}}(\Delta)$. Let $\Delta' \equiv \Delta, x :^{\gamma} A$. Then we have $\Gamma \curlyeqprec \Delta'$ and $\Delta' \vdash x \in \mathcal{E}[\![A]\!]$. By assumption, we obtain $\Delta' \vdash M[\gamma := \gamma, x := x] \equiv M \in \mathcal{E}[\![B]\!]$, and Lemma D.9 yields $M \in \mathrm{SN}$. Thus, $\lambda x :^{\gamma} A.\ M \in \mathrm{SN}$.

▷ Case ($\mathcal{K}$-$\to$). Then we have $K \equiv K'[[-]N]$ for some $\Delta \vdash N \in \mathcal{E}[\![A]\!]$ and $\Delta \vdash K' \in \mathcal{K}[\![B]\!]$. By assumption we have $\Delta \vdash M[\gamma := \mathrm{pos}(\Delta), x := N] \in \mathcal{E}[\![B]\!]$, and hence $K'[M[\gamma := \mathrm{pos}(\Delta), x := N]] \in \mathrm{SN}$. By Lemma D.10 we see $K'[(\lambda x :^{\gamma} A.\ M)N] \in \mathrm{SN}$.

Rule $\mathcal{E}$-$\Box$. Suppose $\left(\Delta, \blacktriangleleft^{\delta}\right) \succcurlyeq \Gamma$. Take $\Delta, \blacktriangleleft^{\delta} \vdash K \in \mathcal{K}[\![\Box^{\succeq\gamma_1} A]\!]$ to show $K[\mathbf{quo}\{^{\gamma_2 :\succeq \gamma_1} M\}] \in \mathrm{SN}$. For the last rule of the derivation of K, there are two possibilities:

▷ Case ($\mathcal{K}$-id). We may assume without loss of generality that $\gamma_2 \notin \mathbf{Dom}_{\mathrm{C}}(\Delta)$. Let $\Delta' \equiv \Delta, \blacktriangleleft^{\delta}, \blacktriangleright^{\gamma_2 :\succeq \gamma_1}$. Then we have $\Gamma \preccurlyeq \left(\Delta', \blacktriangleleft^{\delta}\right)$ with $\Delta' \vdash \delta \sqsubseteq \gamma_2$ and $\Delta' \vdash \gamma_1 \preceq \gamma_2$. By assumption we have $\Delta' \vdash M[\gamma_2 := \gamma_2] \equiv M \in \mathcal{E}[\![A]\!]$, and Lemma D.9 yields $M \in \mathrm{SN}$. Thus, $\mathbf{quo}\{^{\gamma_2 :\succeq \gamma_1} M\} \in \mathrm{SN}$.

▷ Case ($\mathcal{K}$-$\Box$). Then we have $K \equiv K'[\mathbf{unq}\{^{\delta}\,[-]\}]$ for some $\Delta \vdash K \in \mathcal{K}[\![A]\!]$ with $\Delta \vdash \delta \sqsubseteq \mathrm{pos}(\Delta)$ and $\Delta \vdash \gamma_1 \preceq \mathrm{pos}(\Delta)$. By assumption we have $\Delta \vdash M[\gamma_2 := \mathrm{pos}(\Delta)] \in \mathcal{E}[\![A]\!]$, and hence $K'[M[\gamma_2 := \mathrm{pos}(\Delta)]] \in \mathrm{SN}$. By Lemma D.10 we see $K'[\mathbf{unq}\{^{\delta}\,\mathbf{quo}\{^{\gamma_2 :\succeq \gamma_1} M\}\}] \in \mathrm{SN}$.

Rule $\mathcal{E}$-$\forall$. Suppose $\Delta \succcurlyeq \Gamma$. Take $\Delta \vdash K \in \mathcal{K}[\![\forall \gamma_2 :\succeq \gamma_1.\ A]\!]$ to show $K[\lambda \gamma_2 :\succeq \gamma_1.\ M] \in \mathrm{SN}$. For the last rule of the derivation of K, there are two possibilities:

▷ Case ($\mathcal{K}$-id). We may assume without loss of generality that $\gamma_2 \notin \mathbf{Dom}_{\mathrm{C}}(\Delta)$. Let $\Delta' \equiv \Delta, \gamma_2 :\succeq \gamma_1$. Then we have $\Gamma \preccurlyeq \Delta'$ with $\Delta' \vdash \gamma_1 \preceq \gamma_2$. By assumption we obtain $\Delta' \vdash M[\gamma_2 := \gamma_2] \equiv M \in \mathcal{E}[\![A]\!]$, and Lemma D.9 yields $M \in \mathrm{SN}$. Thus, $\lambda \gamma_2 :\succeq \gamma_1.\ M \in \mathrm{SN}$.

▷ Case ($\mathcal{K}$-$\forall$). Then we have $K \equiv K'[[-]\gamma]$ for some $\Delta \vdash K' \in \mathcal{K}[\![A[\gamma_2 := \gamma]]\!]$ with $\Delta \vdash \gamma_1 \preceq \gamma$. By assumption we obtain $\Delta \vdash M[\gamma_2 := \gamma] \in \mathcal{E}[\![A[\gamma_2 := \gamma]]\!]$, and hence $K'[M[\gamma_2 := \gamma]] \in \mathrm{SN}$. By Lemma D.10 we see $K'[(\forall \gamma_2 :\succeq \gamma_1.\ M)\gamma] \in \mathrm{SN}$. ◀

▶ **Lemma D.12** (Monotonicity w.r.t. $\preccurlyeq$). *Suppose* $\Gamma \preccurlyeq \Delta$. *If* $\Gamma \vdash M \in \mathcal{E}[\![A]\!]$, *then* $\Delta \vdash M \in \mathcal{E}[\![A]\!]$.

Proof. If $\mathcal{E}$-id is applied, then it can also be applied to x in Δ as $\Delta \succcurlyeq \Gamma$; otherwise, take $\Delta' \succcurlyeq \Delta$ and $\Delta' \vdash K \in \mathcal{K}[\![A]\!]$. Then we have $\Gamma \preccurlyeq \Delta \preccurlyeq \Delta'$, and thus $K[M] \in \mathrm{SN}$ as $\Gamma \vdash M \in \mathcal{E}[\![A]\!]$, which implies $\Delta \vdash M \in \mathcal{E}[\![A]\!]$. ◀

▶ **Lemma D.13** (Monotonicity regarding $\mathcal{C}[\![\Gamma]\!]$). *Suppose* $\Delta \vdash \sigma \in \mathcal{C}[\![\Gamma]\!]$.

1. *It holds that* $\Delta \vdash \mathrm{pos}(\Gamma)\,\sigma \preceq \mathrm{pos}(\Delta)$.
2. *If* $\Gamma \vdash \gamma_1 \preceq \gamma_2$, *then* $\Delta \vdash \gamma_1\,\sigma \preceq \gamma_2\,\sigma$.
3. *If* $\Gamma \vdash \gamma_1 \sqsubseteq \gamma_2$, *then* $\Delta \vdash \gamma_1\,\sigma \sqsubseteq \gamma_2\,\sigma$.

Proof. By induction on the derivation of $\Delta \vdash \sigma \in \mathcal{C}[\![\Gamma]\!]$. ◀

▶ **Lemma D.14.** *Suppose*

$$\begin{array}{c} \vdots \\ \Delta \vdash \sigma \in \mathcal{C}[\![\Gamma]\!] \\ \vdots \\ \Delta' \vdash \sigma \cdot \sigma' \in \mathcal{C}[\![\Gamma, \Gamma']\!] \end{array}$$

If $\Gamma \preccurlyeq \Gamma, \Gamma'$, *then* $\Delta' \preccurlyeq \Delta$.

$$\begin{tikzcd}
\Gamma, \Gamma' \arrow[r, "\sigma \cdot \sigma'"] & \Delta' \\
\Gamma \arrow[u, "\preccurlyeq"] \arrow[r, "\sigma"] & \Delta \arrow[u, dashed, "\preccurlyeq"']
\end{tikzcd}$$

Proof. Check all conditions for $\Delta \preccurlyeq \Delta'$.

To show $\Delta' \vdash \mathrm{pos}(\Delta) \preceq \mathrm{pos}(\Delta')$, we proceed by induction on derivation. If either $\mathcal{C}$-weak or $\mathcal{C}$-$\forall$ is applied, then it follows from the IH; otherwise, we have $\mathrm{pos}(\Gamma,\, \Gamma')\,(\sigma \cdot \sigma') \equiv \mathrm{pos}(\Delta')$. By Lemma D.13, it holds that

- $\Delta' \vdash \mathrm{pos}(\Gamma)\,\sigma \preceq \mathrm{pos}(\Gamma,\, \Gamma')$ and
- $\Delta' \vdash \mathrm{pos}(\Gamma,\, \Gamma')\,\sigma \preceq \mathrm{pos}(\Delta')$,

which implies $\Delta' \vdash \mathrm{pos}(\Delta) \preceq \mathrm{pos}(\Delta')$.

The other conditions are shown by straightforward induction. ◀

▶ **Lemma D.15** (Fundamental property). *Suppose* $\Gamma \vdash M : A$*. If* $\Delta \vdash \sigma \in \mathcal{C}[\![\Gamma]\!]$*, then* $\Delta \vdash M\,\sigma \in \mathcal{E}[\![A\,\sigma]\!]$*.*

Proof. By induction on the derivation of $\Gamma \vdash M : A$. Taking $\Delta \vdash \sigma \in \mathcal{C}[\![\Gamma]\!]$, We analyze the last rule of the derivation:

▷ Case (Var). If $x \notin \mathbf{Dom}_{\mathrm{V}}(\sigma)$, then it follows from Lemma D.12; otherwise, it follows from Lemma D.14.

▷ Case ($\to$-I). Assume

$$\frac{\Gamma,\, x :^{\gamma} B \vdash N : C \qquad \gamma \notin \mathbf{FC}(C)}{\Gamma \vdash \lambda x :^{\gamma} B.\ N : B \to C}$$

Take $\Delta' \succcurlyeq \Delta$ with $\Delta' \vdash P \in \mathcal{E}[\![B\,\sigma]\!]$. Then we have

$$\dfrac{\dfrac{\Delta \vdash \sigma \in \mathcal{C}[\![\Gamma]\!]}{\Delta' \vdash \sigma \in \mathcal{C}[\![\Gamma]\!]}\ \mathcal{C}\text{-weak} \qquad \Delta' \vdash P \in \mathcal{E}[\![B\,\sigma]\!]}{\Delta' \vdash \sigma \cdot [\gamma := \mathrm{pos}(\Delta'),\, x := P] \in \mathcal{C}[\![\Gamma,\, x :^{\gamma} B]\!]}\ \mathcal{C}\text{-}{\to}$$

By the IH, we have

$$\begin{aligned}&\Delta' \vdash N\,(\sigma \cdot [\gamma := \mathrm{pos}(\Delta'),\, x := P])\\ &\quad \equiv (N\,\sigma)[\gamma := \mathrm{pos}(\Delta'),\, x := P] \in \mathcal{E}[\![C\,\sigma]\!],\end{aligned}$$

and by $\mathcal{E}$-$\to$, we see

$$\begin{aligned}&\Delta \vdash \lambda x :^{\gamma} B\,\sigma.\ N\,\sigma\\ &\quad \equiv (\lambda x :^{\gamma} B.\ N)\,\sigma \in \mathcal{E}[\![(B \to C)\,\sigma]\!].\end{aligned}$$

▷ Case ($\to$-E). Assume

$$\frac{\Gamma \vdash N : B \to C \qquad \Gamma \vdash P : B}{\Gamma \vdash N P : C}$$

Take $\Delta' \succcurlyeq \Delta$ with $\Delta \vdash K \in \mathcal{K}[\![C\,\sigma]\!]$. Then we have

$$\dfrac{\dfrac{\Delta \vdash \sigma \in \mathcal{C}[\![\Gamma]\!]}{\Delta' \vdash \sigma \in \mathcal{C}[\![\Gamma]\!]}\ \mathcal{C}\text{-weak} \qquad \Gamma \vdash P : B}{\Delta' \vdash P\,\sigma \in \mathcal{E}[\![B\,\sigma]\!]}\ \mathrm{IH}$$

and

$$\frac{\Delta' \vdash K \in \mathcal{K}[\![C\,\sigma]\!] \qquad \Delta' \vdash P\,\sigma \in \mathcal{E}[\![B\,\sigma]\!]}{\Delta' \vdash K[[-](P\,\sigma)] \in \mathcal{E}[\![(B \to C)\,\sigma]\!]}\ \mathcal{K}\text{-}{\to}$$

Applying the IH to N yields $\Delta \vdash N\,\sigma \in \mathcal{E}[\![(B \to C)\,\sigma]\!]$, so that $K[(N P)\,\sigma] \in \mathrm{SN}$, implying $\Delta \vdash (N P)\,\sigma \in \mathcal{E}[\![C\,\sigma]\!]$.

▷ Case ($\square$-I). Assume

$$\frac{\Gamma,\ \blacktriangleright^{\gamma_2 : \succeq \gamma_1} \vdash N : B \qquad \gamma_2 \notin \mathbf{FC}(B)}{\Gamma \vdash \mathbf{quo}\{^{\gamma_2 : \succeq \gamma_1}\, N\} : \square^{\succeq \gamma_1} B}$$

Take $\Delta',\ \blacktriangleleft^{\delta} \succcurlyeq \Delta$ with $\Delta' \vdash \delta \sqsubseteq \mathrm{pos}(\Delta')$ and $\Delta' \vdash \gamma_1\,\sigma \preceq \mathrm{pos}(\Delta')$. Then we have

$$\dfrac{\dfrac{\Delta,\ \blacktriangleleft^{\delta} \vdash \sigma \in \mathcal{C}[\![\Gamma]\!]}{\Delta',\ \blacktriangleleft^{\delta} \vdash \sigma \in \mathcal{C}[\![\Gamma]\!]}\ \mathcal{C}\text{-weak} \qquad \Delta' \vdash \gamma_1\,\sigma \preceq \mathrm{pos}(\Delta')}{\Delta' \vdash \sigma \cdot [\gamma_2 \coloneqq \mathrm{pos}(\Delta')] \in \mathcal{C}[\![\Gamma,\ \blacktriangleright^{\gamma_2 : \succeq \gamma_1}]\!]}\ \mathcal{C}\text{-}\square$$

By the IH, we have

$$\begin{aligned} &\Delta' \vdash N\,(\sigma \cdot [\gamma_2 \coloneqq \mathrm{pos}(\Delta')]) \\ &\quad \equiv (N\,\sigma)[\gamma_2 \coloneqq \mathrm{pos}(\Delta')] \in \mathcal{E}[\![B\,\sigma]\!], \end{aligned}$$

and by $\mathcal{E}$-$\square$, we see

$$\begin{aligned} &\Delta \vdash \mathbf{quo}\{^{\gamma_2 : \succeq \gamma_1\,\sigma}\, N\,\sigma\} \\ &\quad \equiv \mathbf{quo}\{^{\gamma_2 : \succeq \gamma_1}\, N\}\,\sigma \in \mathcal{E}[\![\square^{\succeq \gamma_1} B\,\sigma]\!]. \end{aligned}$$

▷ Case ($\square$-E). Assume

$$\frac{\Gamma,\ \blacktriangleleft^{\gamma} \vdash N : \square^{\succeq \gamma_1} B \qquad \Gamma \vdash \gamma_1 \preceq \mathrm{pos}(\Gamma)}{\Gamma \vdash \mathbf{unq}\{^{\gamma}\, N\} : B}$$

Take $\Delta' \succcurlyeq \Delta$ with $\Delta' \vdash K \in \mathcal{K}[\![B\,\sigma]\!]$. Using Lemma D.13 we have

$$\begin{array}{c} \begin{array}{c} \Gamma \vdash \gamma \sqsubseteq \mathrm{pos}(\Gamma) \\ \hdashline \Delta' \vdash \gamma\,\sigma \sqsubseteq \mathrm{pos}(\Gamma)\,\sigma \end{array} \qquad \begin{array}{c} \\ \hdashline \Delta' \vdash \mathrm{pos}(\Gamma)\,\sigma \preceq \mathrm{pos}(\Delta') \end{array} \\ \hdashline \Delta' \vdash \gamma\,\sigma \sqsubseteq \mathrm{pos}(\Delta') \end{array}$$

$$\begin{array}{c} \begin{array}{c} \Gamma \vdash \gamma_1 \preceq \mathrm{pos}(\Gamma) \\ \hdashline \Delta' \vdash \gamma\,\sigma \sqsubseteq \mathrm{pos}(\Gamma)\,\sigma \end{array} \qquad \begin{array}{c} \\ \hdashline \Delta' \vdash \mathrm{pos}(\Gamma)\,\sigma \preceq \mathrm{pos}(\Delta') \end{array} \\ \hdashline \Delta' \vdash \gamma_1\,\sigma \preceq \mathrm{pos}(\Delta') \end{array}$$

and $\Delta',\ \blacktriangleleft^{\gamma\,\sigma} \vdash K[\mathbf{unq}\{^{\gamma\,\sigma}\,[-]\}] \in \mathcal{K}[\![\square^{\succeq \gamma_1} B\,\sigma]\!]$ by $\mathcal{K}$-$\square$. In addition, we have

$$\begin{array}{c} \dfrac{\dfrac{\Delta \vdash \sigma \in \mathcal{C}[\![\Gamma]\!]}{\Delta' \vdash \sigma \in \mathcal{C}[\![\Gamma]\!]}\ \mathcal{C}\text{-weak}}{\Delta',\ \blacktriangleleft^{\gamma\,\sigma} \vdash \sigma \in \mathcal{C}[\![\Gamma,\ \blacktriangleleft^{\gamma}]\!]}\ \mathcal{C}\text{-}\blacktriangleleft \qquad \Gamma,\ \blacktriangleleft^{\gamma} \vdash N : \square^{\succeq \gamma_1} B \\ \hdashline \Delta',\ \blacktriangleleft^{\gamma\,\sigma} \vdash N\,\sigma \in \mathcal{E}[\![\square^{\succeq \gamma_1} B\,\sigma]\!] \end{array}\ \mathrm{IH}$$

and therefore $K[\mathbf{unq}\{^{\gamma\,\sigma}\, N\,\sigma\}] \equiv K[(\mathbf{unq}\{^{\gamma}\, N\})\,\sigma] \in \mathrm{SN}$, implying $\Delta \vdash (\mathbf{unq}\{^{\gamma}\, N\})\,\sigma \in \mathcal{E}[\![B\,\sigma]\!]$.

▷ Case (∀-I). Assume

$$\frac{\Gamma,\ \gamma_2 :\succeq \gamma_1 \vdash N : B}{\Gamma \vdash \lambda\gamma_2 :\succeq \gamma_1.\ N : \forall\gamma_2 :\succeq \gamma_1.\ B}$$

Take $\Delta' \succcurlyeq \Delta$ with $\Delta' \vdash \gamma_1\,\sigma \preceq \gamma$. Then we have

$$\cfrac{\cfrac{\Delta \vdash \sigma \in \mathcal{C}\llbracket\Gamma\rrbracket}{\Delta' \vdash \sigma \in \mathcal{C}\llbracket\Gamma\rrbracket}\ \mathcal{C}\text{-weak} \qquad \Delta' \vdash \gamma_1\,\sigma \preceq \gamma}{\Delta' \vdash \sigma \cdot [\gamma_2 \coloneqq \gamma] \in \mathcal{C}\llbracket\Gamma,\ \gamma_2 :\succeq \gamma_1\rrbracket}\ \mathcal{C}\text{-}\forall$$

By the IH, we have $\Delta' \vdash (N\,\sigma)[\gamma_2 \coloneqq \gamma] \in \mathcal{E}\llbracket B\,\sigma\rrbracket$, and by $\mathcal{E}$-∀ we see

$$\begin{aligned}&\Delta \vdash \lambda\gamma_2 :\succeq \gamma_1\,\sigma.\ N\,\sigma\\ &\quad\equiv (\lambda\gamma_2 :\succeq \gamma_1.\ N)\,\sigma \in \mathcal{E}\llbracket(\forall\gamma_2 :\succeq \gamma_1.\ B)\,\sigma\rrbracket.\end{aligned}$$

▷ Case (∀-E). Assume

$$\frac{\Gamma \vdash N : \forall\gamma_2 :\succeq \gamma_1.\ B \qquad \Gamma \vdash \gamma_1 \preceq \gamma}{\Gamma \vdash N\gamma : B[\gamma_2 \coloneqq \gamma]}$$

Take $\Delta' \succcurlyeq \Delta$ with $\Delta \vdash K \in \mathcal{K}\llbracket(B[\gamma_2 \coloneqq \gamma])\,\sigma\rrbracket$. Using Lemma D.13 we have

$$\cfrac{\cfrac{\Gamma \vdash \gamma_1 \preceq \gamma}{\Delta' \vdash \gamma_1\,\sigma \preceq \gamma\,\sigma} \qquad \Delta' \vdash K \in \mathcal{K}\llbracket(B[\gamma_2 \coloneqq \gamma])\,\sigma\rrbracket}{\Delta' \vdash K[[-](\gamma\,\sigma)] \in \mathcal{C}\llbracket(\forall\gamma_2 :\succeq \gamma_1.\ B)\,\sigma\rrbracket}\ \mathcal{C}\text{-}\forall$$

Applying the IH to N yields $\Delta' \vdash N\,\sigma \in \mathcal{E}\llbracket(\forall\gamma_2 :\succeq \gamma_1.\ B)\,\sigma\rrbracket$, so that $K[(N\gamma)\,\sigma] \in \mathrm{SN}$, implying $\Delta \vdash (N\gamma)\,\sigma \in \mathcal{E}\llbracket(B[\gamma_2 \coloneqq \gamma])\,\sigma\rrbracket$. ◀

▶ **Theorem 6.8** (Strong Normalization (On page 12)). *If $\Gamma^{\gamma} \vdash M : A$, then M is strongly normalizing with respect to $\Rightarrow^{\gamma}_{\beta}$.*

Proof. Given $\Gamma \vdash M : A$. Taking $\Gamma \vdash \emptyset \in \mathcal{C}\llbracket\Gamma\rrbracket$ yields $\Gamma \vdash M \in \mathcal{E}\llbracket A\rrbracket$ by Lemma D.15; thus $M \in \mathrm{SN}$ by Lemma D.9. ◀

▶ **Theorem 6.9** (Confluence (On page 12)). *If $\Gamma^{\gamma} \vdash M_1 : A$, $M_1 \Rightarrow^{\gamma}_{\beta*} M_2$ and $M_1 \Rightarrow^{\gamma}_{\beta*} M_3$, then there exists M_4 such that $M_2 \Rightarrow^{\gamma}_{\beta*} M_4$ and $M_3 \Rightarrow^{\gamma}_{\beta*} M_4$.*

Proof. From Newman's lemma [31] and Theorem 6.8, we need only to prove weak confluence. It is done by induction on β, which is straightforward because there are no critical pairs. ◀

▶ **Lemma D.16.** *Suppose $\Gamma \vdash M : A$. If M is β-normal and neutral, then there exists some $x :^{\gamma} B \in \Gamma$ such that A is a subformula of B.*

Proof. By induction on derivation. For the last rule of the derivation, there are four possibilities:

▷ Case (Var). Obvious.

▷ Case (→-E). Assume

$$\frac{\Gamma \vdash N : B \to C \qquad \Gamma \vdash P : B}{\Gamma \vdash NP : C}$$

Since NP is β-normal, N is β-normal and neutral. By the IH there exists some $x :^{\delta} D \in \Gamma$ such that $B \to C$ is a subformula of D. Then C is also a subformula of D, hence $x :^{\delta} D$ meets the condition.

▷ Case ($\Box$-E). Assume

$$\frac{\Gamma, \blacktriangleleft^{\gamma} \vdash N : \Box^{\succeq \gamma_1} B \qquad \Gamma \vdash \gamma_1 \preceq \mathrm{pos}(\Gamma)}{\Gamma \vdash \mathbf{unq}\{^{\gamma} N\} : B}$$

Since $\mathbf{unq}\{^{\gamma} N\}$ is β-normal, N is β-normal and neutral. By the IH there exists some $x :^{\delta} D \in \Gamma, \blacktriangleleft^{\gamma}$ such that $\Box^{\succeq \gamma_1} B$ is a subformula of D. Then B is also a subformula of D, hence $x :^{\delta} D$ meets the condition.

▷ Case (∀-E). Assume

$$\frac{\Gamma \vdash N : \forall \gamma_2 :\succeq \gamma_1.\, B \qquad \Gamma \vdash \gamma_1 \preceq \gamma}{\Gamma \vdash N\gamma : B[\gamma_2 := \gamma]}$$

Since $N\gamma$ is β-normal, N is β-normal and neutral. By the IH there exists some $x :^{\delta} D \in \Gamma$ such that $\forall \gamma_2 :\succeq \gamma_1.\, B$ is a subformula of D. Then $B[\gamma_2 := \gamma]$ is also a subformula of D, hence $x :^{\delta} D$ meets the condition. Notice that classifier renaming $[\gamma_2 := \gamma]$ here is allowed in the definition of subformula. ◀

▶ **Theorem 6.12** (Canonicity (On page 12)). *If a term is well-typed, closed regarding term variable, and β-normal, then it is canonical.*

Proof. If not canonical, by Lemma D.16 it contains a free variable, which contradicts the assumption. ◀

▶ **Theorem 6.13** (Subformula Property (On page 12)). *Suppose $\Gamma^{\gamma} \vdash M : A$. If M is normal with respect to $\Rightarrow^{\gamma}_{\beta}$, then any subterm of M satisfies at least one of the following:*

1. *Its type is a subformula of A;*
2. *Its type is a subformula of B for some $x :^{\delta} B \in \Gamma$.*

Proof. By induction on derivation. Since the term M itself clearly satisfies (1), it suffices to check condition for proper subterms.

▷ Case (Var). No proper subterm exists.

▷ Case (→-I). Assume

$$\frac{\Gamma, y :^{\gamma} B \vdash N : C \qquad \gamma \notin \mathbf{FC}(C)}{\Gamma \vdash \lambda y :^{\gamma} B.\, N : B \to C}$$

Since all proper subterms of $\lambda y :^{\gamma} B.\, N$ are a subterm of N, by the IH there are three possibilities for their types:

▷ Subcase (a). A subformula of C. Then it is also a subformula of $B \to C$, so (1) holds.

▷ Subcase (b). A subformula of B. This is also the case (1).

▷ Subcase (c). A subformula of D for some $x :^{\delta} D \in \Gamma$. Yields (2).

▷ Case ($\to$-E). Assume

$$\frac{\Gamma \vdash N : B \to C \qquad \Gamma \vdash P : B}{\Gamma \vdash NP : C}$$

Then N is β-normal and neutral, so applying Lemma D.16 to N, we see that there exists some $x :^{\delta} D \in \Gamma$ such that $B \to C$ is a subformula of D. Together with the IH, we see that (2) holds for any proper subterm of NP.

▷ Case ($\Box$-I). Follows from the IH.

▷ Case ($\Box$-E). Similar to the rule $\to$-E.

▷ Case ($\forall$-I). Follows from the IH.

▷ Case ($\forall$-E). Similar to the rule $\to$-E. ◀

E Full Definitions and Proofs for Section 7

E.1 Semantical Comparison

▶ **Definition E.1.** *We define two functions* $|-| : \mathcal{L}_{!} \to \mathcal{L}_{\Box}$ *and* $(-)^{\succeq !} : \mathcal{L}_{\Box} \to \mathcal{L}_{!}$ *as follows:*

$$|p| = p \qquad (p)^{\succeq !} = p$$
$$|A \to B| = |A| \to |B| \qquad (A \to B)^{\succeq !} = (A)^{\succeq !} \to (B)^{\succeq !}$$
$$|\Box^{\succeq !} A| = \Box |A| \qquad (\Box A)^{\succeq !} = \Box^{\succeq !}(A)^{\succeq !}$$

▶ **Lemma 7.2** (Stabilization (On page 13)). *Given a* **CS4***-model* $M = \langle W, \preceq, R, V \rangle$*. Define* $\sqsubseteq$ *as* $(\preceq ; R)$*. Then* $M^{*} = \langle W, \preceq, \sqsubseteq, V \rangle$ *is a stable* **CS4***-model.*

Proof. The transitivity of $\sqsubseteq$ follows from the left-persistency of R, and the stability follows from reflexivity of R. ◀

▶ **Lemma E.2.** *Given a* **CS4***-model* M*. For any* $A \in \mathcal{L}_{\Box}$*, the following are equivalent:*

1. $M, w \vDash_{\mathbf{CS4}} A$;
2. $M^{*}, w \vDash_{\mathbf{CS4}} A$.

Proof. By induction on A. Here we check the case $A \equiv \Box B$:

$$\begin{aligned} & M, w \vDash_{\mathbf{CS4}} \Box B \\ \iff & \forall v \succeq w.\ \forall u \in R(v).\ \big(M,\ \ w \vDash_{\mathbf{CS4}} \Box B\big) \\ \iff & \forall v \succeq w.\ \forall u \sqsupseteq v.\ \ \ \big(M,\ \ w \vDash_{\mathbf{CS4}} \Box B\big) && (*) \\ \iff & \forall v \succeq w.\ \forall u \sqsupseteq v.\ \ \ \big(M^{*}, w \vDash_{\mathbf{CS4}} \Box B\big) && \text{(by IH)} \\ \iff & M^{*}, w \vDash_{\mathbf{CS4}} \Box B \end{aligned}$$

where the left-to-right direction of $(*)$ follows from reflexivity of $\preceq$, and the converse follows from transitivity of $\preceq$. ◀

▶ **Lemma 7.3** (Root-Extension (On page 13)). *Given a stable* **CS4***-model* $M = \langle W, \preceq, \sqsubseteq, V \rangle$*. Define* $M_{!} = \langle W_{!}, \preceq_{!}, \sqsubseteq_{!}, V_{!} \rangle$ *as follows:*

- $W_{\mathbf{!}} = W \amalg \{\mathbf{!}\}$;
- $w \preceq_{\mathbf{!}} v \iff w = \mathbf{!}$ *or* $w \preceq v$;
- $w \sqsubseteq_{\mathbf{!}} v \iff w = \mathbf{!}$ *or* $w \sqsubseteq v$;
- $w \in V_{\mathbf{!}}(p) \iff V(p) = W$ *if* $w = \mathbf{!}$, *and* $w \in V(p)$ *otherwise.*

Then $M_{!}$ *is a stable* **CS4**-*model with a root* **!**, *namely, a* **BML**-*structure.*

Proof. Straightforward. ◀

▶ **Lemma E.3.** *Given a stable* **CS4**-*model* M. *For any* $A \in \mathcal{L}_{\Box}$, *the following are equivalent:*
1. $M, w \vDash_{\mathbf{CS4}} A$;
2. $M_{\mathbf{!}}, w \vDash_{\mathbf{CS4}} A$.

Proof. By induction on A. ◀

▶ **Lemma 7.4** (One-Point Model (On page 13)). *Given a* **BML**-*structure* M. *Define* M_* *as*

$$\langle \{*\}, \{\langle *, * \rangle\}, \{* \mapsto M\} \rangle.$$

Then M_* *is a* **BML**-*model.*

Proof. Obvious. ◀

▶ **Lemma E.4.** *Given a* **BML**-*structure* M. *Define a* $*$-*assignment* **!** *for the one-point mode* M_* *as* $\mathbf{!} \mapsto \mathbf{!}$. *Then, for any* $A \in \mathcal{L}_{\Box}$, *the following are equivalent:*
1. $M, w \vDash_{\mathbf{CS4}} A$;
2. $M_*, *, w \Vdash^{\mathbf{!}} A^{\succeq \mathbf{!}}$.

Proof. By induction on A. There are three cases:

▷ Case ($A \equiv \alpha$).

$$\begin{aligned} & M, w \vDash_{\mathbf{CS4}} \alpha \\ \iff & w \in V(\alpha) \\ \iff & M_*, *, w \Vdash^{\mathbf{!}} \alpha^{\succeq \mathbf{!}} \end{aligned}$$

▷ Case ($A \equiv B \to C$).

$$\begin{aligned} & M, w \vDash_{\mathbf{CS4}} B \to C \\ \iff & \forall v \succeq w.\ \big(M, \quad v \vDash_{\mathbf{CS4}} B \implies M, \quad v \vDash_{\mathbf{CS4}} C\big) \\ \iff & \forall v \succeq w.\ \big(M_*, *, v \Vdash^{\mathbf{!}} \quad B^{\succeq \mathbf{!}} \implies M_*, *, v \Vdash^{\mathbf{!}} \quad C^{\succeq \mathbf{!}}\big) && \text{(by IH)} \\ \iff & M_*, *, w \Vdash^{\mathbf{!}} (B \to C)^{\succeq \mathbf{!}} \end{aligned}$$

▷ Case ($A \equiv \Box B$).

$$\begin{aligned} & M, w \vDash_{\mathbf{CS4}} \Box B \\ \iff & \forall v \succeq w.\ \forall u \sqsupseteq v.\ \big(M, \quad u \vDash_{\mathbf{CS4}} B\big) \\ \iff & \forall u \sqsupseteq w.\ \qquad \big(M, \quad u \vDash_{\mathbf{CS4}} B\big) && (\sqsubseteq \text{ is stable}) \\ \iff & \forall u \sqsupseteq w.\ \qquad \big(M_*, *, u \Vdash^{\mathbf{!}} \quad B^{\succeq \mathbf{!}}\big) && \text{(by IH)} \\ \iff & M_*, *, w \Vdash^{\mathbf{!}} \Box^{\succeq \mathbf{!}} B^{\succeq \mathbf{!}} \end{aligned}$$

◀

▶ **Theorem 7.5** (On page 13). *Given a* **CS4**-*model* $M = \langle W, \preceq, R, V \rangle$. *Define* $\mathfrak{M}$ *as* $(M^*)_{!*}$ *and a* $*$-*assignment* **!** *for* $\mathfrak{M}$ *as* $\mathbf{!} \mapsto \mathbf{!}$. *Then* $\mathfrak{M}$ *is a* **BML**-*model, and for any* $A \in \mathcal{L}_{\Box}$, *the following are equivalent:*

- $M,\ w \vDash_{\mathbf{CS4}} A$;
- $\mathfrak{M},\ *,\ w \Vdash^{!} (A)^{\succeq !}$.

Proof. It follows that $\mathfrak{M}$ is a **BML**-model from Lemmas 7.2–7.4, and for any $A \in \mathcal{L}_{\Box}$, we have

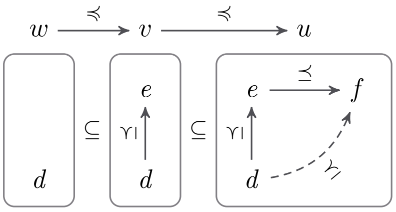
$$\begin{aligned}
& M, && w \vDash_{\mathbf{CS4}} A && \\
\iff & M^{*}, && w \vDash_{\mathbf{CS4}} A && \text{(by Lemma E.2)}\\
\iff & (M^{*})_{!}, && w \vDash_{\mathbf{CS4}} A && \text{(by Lemma E.3)}\\
\iff & \mathfrak{M},\ *, && w \Vdash^{!} \quad A^{\succeq !} && \text{(by Lemma E.4)}
\end{aligned}$$

◀

E.2 Flattening

▶ **Lemma 7.7** (On page 14). *$\mathfrak{M}_+$ is a* **CS4***-model.*

Proof. Most of the conditions are straightforward. The transitivity of $\preceq_+$ follows from that of $\preceq_u$: if $\langle w,\, d\rangle \preceq_+ \langle v,\, e\rangle$ and $\langle v,\, e\rangle \preceq_+ \langle u,\, f\rangle$, then $\langle w,\, d\rangle \preceq_+ \langle u,\, f\rangle$:

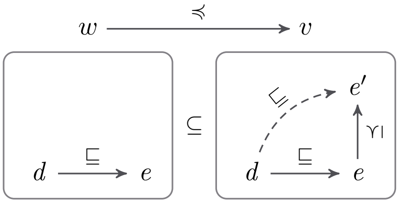


in $\mathfrak{M}$

$$\begin{tikzcd}
& \langle v,\, e\rangle \arrow[r, "\preceq_+"] & \langle u, f\rangle \\
\langle w,\, d\rangle \arrow[ur, "\preceq_\times"] \arrow[urr, dashed, bend right, "\preceq_+"'] & &
\end{tikzcd}$$

in $\mathfrak{M}_+$

Left-persistency follows from right-stability: if $\langle w,\, d\rangle\ R_+\ \langle w,\, e\rangle \preceq_+ \langle v,\, e'\rangle$, then $\langle w,\, d\rangle \preceq_+ \langle v,\, d\rangle\ R_+\ \langle v,\, e'\rangle$:



in $\mathfrak{M}$

$$\begin{tikzcd}
& & \langle v,\, d\rangle \arrow[r, dashed, "R_+"] & \langle v,\, e'\rangle \\
\langle w,\, d\rangle \arrow[urr, dashed, "\preceq_+"] \arrow[r, "R_+"'] & \langle w,\, e\rangle \arrow[urr, "\preceq_+"'] & &
\end{tikzcd}$$

in $\mathfrak{M}_+$

◀

▶ **Theorem 7.8** (On page 14). *Given a* **BML***-model $\mathfrak{M}$. For any $A \in \mathcal{L}_{!}$, the following are equivalent:*

- $\mathfrak{M},\ w,\ d \Vdash^{\rho} A$;
- $\mathfrak{M}_+,\ \langle w,\, d\rangle \vDash_{\mathbf{CS4}} |A|$.

Proof. By induction on A. There are three cases:

▷ Case ($A \equiv \alpha$).

$$\begin{aligned} & \mathfrak{M},\, w,\, d \Vdash^{\rho} \alpha \\ \iff & \forall v \succcurlyeq w.\ d \in V_v(\alpha) \\ \iff & d \in V_w(\alpha) && (V_w(\alpha) \text{ is increasing}) \\ \iff & \langle w,\, d\rangle \in V_+(\alpha) \\ \iff & \mathfrak{M}_+,\, \langle w,\, d\rangle \vDash_{\mathbf{CS4}} |\alpha| \end{aligned}$$

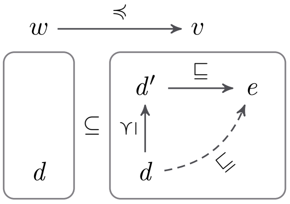
▷ Case ($A \equiv B \to C$).

$$\begin{aligned} & \mathfrak{M},\, w,\, d \Vdash^{\rho} B \to C \\ \iff & \forall v \succcurlyeq w.\ \forall e \succeq_v d.\ \begin{cases} \mathfrak{M},\, v,\, e \Vdash^{\rho} B \\ \quad \Longrightarrow \mathfrak{M},\, v,\, e \Vdash^{\rho} C \end{cases} \\ \iff & \forall v \succcurlyeq w.\ \forall e \succeq_v d.\ \begin{cases} \mathfrak{M}_+,\, \langle v,\, e\rangle \vDash_{\mathbf{CS4}} |B| \\ \quad \Longrightarrow \mathfrak{M}_+,\, \langle v,\, e\rangle \vDash_{\mathbf{CS4}} |C| \end{cases} && \text{(IH)} \\ \iff & \forall \langle v,\, e\rangle \succeq_+ \langle w,\, d\rangle.\ \begin{cases} \mathfrak{M}_+,\, \langle v,\, e\rangle \vDash_{\mathbf{CS4}} |B| \\ \quad \Longrightarrow \mathfrak{M}_+,\, \langle v,\, e\rangle \vDash_{\mathbf{CS4}} |C| \end{cases} \\ \iff & \mathfrak{M}_+,\, \langle w,\, d\rangle \vDash_{\mathbf{CS4}} |B \to C| \end{aligned}$$

▷ Case ($A \equiv \Box^{\succeq\,!} B$).

$$\begin{aligned} & \mathfrak{M},\, w,\, d \Vdash^{\rho} \Box^{\succeq\,!} B \\ \iff & \forall v \succcurlyeq w.\ \forall e \sqsupseteq_v d.\ (\mathfrak{M},\, v,\, e \Vdash^{\rho} B) \\ \iff & \forall v \succcurlyeq w.\ \forall e \sqsupseteq_v d.\ \Big(\mathfrak{M}_+,\, \langle v,\, e\rangle \vDash_{\mathbf{CS4}} |B|\Big) && \text{(IH)} \\ \iff & \forall \langle v,\, d'\rangle \succeq_+ \langle w,\, d\rangle. \\ & \quad \forall \langle v,\, e\rangle \in R_+\, (\langle v,\, d'\rangle).\ \Big(\mathfrak{M}_+,\, \langle v,\, e\rangle \vDash_{\mathbf{CS4}} |B|\Big) && (\dagger) \\ \iff & \mathfrak{M}_+,\, \langle w,\, d\rangle \vDash_{\mathbf{CS4}} \big|\Box^{\succeq\,!} B\big| \end{aligned}$$

where the left-to-right direction of (†) follows from left-stability:



in $\mathfrak{M}$

$$\begin{tikzcd} & \langle v,\, d'\rangle \arrow[r, "R_+"] & \langle v,\, e\rangle \\ \langle w,\, d\rangle \arrow[ur, "\succeq^{\times}"] & & \end{tikzcd}$$

in $\mathfrak{M}_+$

◀

▶ **Theorem 7.9** (On page 14). *The $\mathcal{L}_{\Box}$-fragment of* **CS4** *is isomorphic to the $\mathcal{L}_{!}$-fragment of* **BML** *up to logical equivalence.*

Proof. Follows from Theorems 7.5 and 7.8, where $|-|$ and $(-)^{\succeq\,!}$ are the isomorphisms. ◀

E.3 Proof-Theoretic Comparison

Davies and Pfenning [7] provided a modal lambda-calculus that corresponds to their Kripke-style natural-deduction proof system. For convenience, we call it $\lambda^{\Box}$ in this paper. Figure 12 provides the definition of $\lambda^{\Box}$. By forgetting proof terms, we obtain natural-deduction proof system.

Variables	$x,\ y$		
Types	$A,\ B$	$::=$	$p \mid A \to B \mid \Box A$
Terms	$M,\ N$	$::=$	$x \mid \lambda x^{A}.\ M \mid MN$
		$\mid$	$\mathbf{box}\,\{M\} \mid \mathbf{unbox}_k\{M\}$
Context	$\Gamma,\ \Delta$	$::=$	$\varepsilon \mid \Gamma,\ x : A$
Context Stack	Ψ	$::=$	$\Gamma \mid \Psi\ ;\Gamma$

Var

$$\frac{x : A \in \Gamma}{\Psi\ ;\Gamma \vdash_{\mathbf{S4}} x : A}$$

$\to$-I

$$\frac{\Psi\ ;\Gamma,\ x : A \vdash_{\mathbf{S4}} M : B}{\Gamma \vdash_{\mathbf{S4}} \lambda x^{A}.\ M : A \to B}$$

$\to$-E

$$\frac{\Gamma \vdash_{\mathbf{S4}} M : A \to B \qquad \Gamma \vdash_{\mathbf{S4}} N : A}{\Gamma \vdash_{\mathbf{S4}} MN : B}$$

$\Box$-I

$$\frac{\Psi\ ;\varepsilon \vdash_{\mathbf{S4}} M : A}{\Psi \vdash_{\mathbf{S4}} \mathbf{box}\,\{M\} : \Box A}$$

$\Box$-E

$$\frac{\Psi \vdash_{\mathbf{S4}} M : \Box A}{\Psi\ ;\Delta_1\ ;\cdots;\Delta_k \vdash_{\mathbf{S4}} \mathbf{unbox}_k\{M\} : A}$$

Figure 12 Syntax and Typing Rules of $\lambda^{\Box}$

$\lambda^{\Box}$ can be considered as restricted version of our lambda-calculus, where quoted code is always closed. This means that a box type $\Box A$ corresponds to a bounded modal type with an initial classifier $\Box^{\succeq \mathbf{!}} A$. The whole definition of the translation is provided in Figure 13.

The term translation $(M)^{\succeq \mathbf{!}}_{\overrightarrow{\gamma}}$ carries a sequence of classifiers $\overrightarrow{\gamma}$, which represents positions for each past stage. The context translation judgment $\Gamma \rightsquigarrow \Gamma/\overrightarrow{\gamma}$ states that Γ can be translated to Γ where positions of past states are $\overrightarrow{\gamma}$. We can prove this translation preserves typeability.

▶ **Lemma E.5.** *$\Gamma \vdash \mathbf{!} \preceq \gamma$ and $\Gamma \vdash \mathbf{!} \sqsubseteq \gamma$ hold as long as $\gamma \in \mathbf{Dom}_{\mathrm{C}}(\Gamma)$.*

Proof. By induction on the derivation of $\Gamma \vdash \mathbf{!} \preceq \gamma$ and $\Gamma \vdash \mathbf{!} \sqsubseteq \gamma$. ◀

Theorem 7.10 can be proved by translating $\lambda^{\Box}$ terms to terms of our lambda-calculus. Namely, it is a corollary of the following theorem.

▶ **Theorem E.6.** *If $\Psi \vdash_{\mathbf{S4}} M : A$ and $\Psi \rightsquigarrow \Gamma/\overrightarrow{\gamma}$, then $\Gamma \vdash (M)^{\succeq \mathbf{!}}_{\overrightarrow{\gamma}} : (A)^{\succeq \mathbf{!}}$ holds.*

Proof. By induction on the derivation of $\Psi \vdash_{\mathbf{S4}} M : A$. We demonstrate the case of $\Box$-E.

Case $\Box$-E: We have a derivation

$$\frac{\Psi \vdash_{\mathbf{S4}} M' : \Box A}{\Psi\ ;\Gamma_1\ ;\cdots;\Gamma_k \vdash_{\mathbf{S4}} \mathbf{unbox}_k\{M'\} : A}\ \Box\text{-E}$$

Decomposing $\overrightarrow{\gamma}$ to $\overrightarrow{\gamma}',\ \delta_0,\ \ldots,\ \delta_k$, we derive $\Psi \rightsquigarrow \Delta,\ \blacktriangleleft^{\delta_0}/\overrightarrow{\gamma}',\ \delta_0$ from $\Psi\ ;\Gamma_1\ ;\cdots;\Gamma_k \rightsquigarrow \Delta/\overrightarrow{\gamma}',\ \delta_0,\ \ldots,\ \delta_k$. Then we can apply the induction hypothesis, and get $\Delta,\ \blacktriangleleft^{\delta_0} \vdash (M')^{\succeq \mathbf{!}}_{\overrightarrow{\gamma}',\,\delta_0} : \Box^{\succeq \mathbf{!}}(A)^{\succeq \mathbf{!}}$. We apply $\Box$-E to derive $\Delta \vdash \mathbf{unq}\{^{\delta_0}(M')^{\succeq \mathbf{!}}_{\overrightarrow{\gamma}',\,\delta_0}\} : (A)^{\succeq \mathbf{!}}$, which is what we want. ◀

Terms $(M)^{\succeq \mathbf{!}}_{\overrightarrow{\gamma}}$

$$
\begin{aligned}
(x)^{\succeq \mathbf{!}}_{\overrightarrow{\gamma}} &= x \\
(\lambda x^A.\ M)^{\succeq \mathbf{!}}_{\overrightarrow{\gamma},\,\delta_1} &= \lambda x :^{\delta_2} (A)^{\succeq \mathbf{!}}.\,(M)^{\succeq \mathbf{!}}_{\overrightarrow{\gamma},\,\delta_2} \\
&\ \text{where } \delta_2 \text{ is fresh} \\
(MN)^{\succeq \mathbf{!}}_{\overrightarrow{\gamma}} &= (M)^{\succeq \mathbf{!}}_{\overrightarrow{\gamma}}(N)^{\succeq \mathbf{!}}_{\overrightarrow{\gamma}} \\
(\mathbf{box}\,\{M\})^{\succeq \mathbf{!}}_{\overrightarrow{\gamma}} &= \mathbf{quo}\{^{\delta : \succeq \mathbf{!}}(M)^{\succeq \mathbf{!}}_{\overrightarrow{\gamma},\,\delta}\} \quad \text{where } \delta \text{ is a fresh classifier} \\
(\mathbf{unbox}_k\{M\})^{\succeq \mathbf{!}}_{\overrightarrow{\gamma},\,\delta_0,\,\dots,\,\delta_k} &= \mathbf{unq}\{^{\delta_0}(M)^{\succeq \mathbf{!}}_{\overrightarrow{\gamma},\,\delta_0}\}
\end{aligned}
$$

Contexts $\Psi \rightsquigarrow \Gamma / \overrightarrow{\gamma}$

$$
\frac{}{\varepsilon \rightsquigarrow \varepsilon / \mathbf{!}}
\qquad
\frac{\Psi\,;\Gamma \rightsquigarrow \Gamma / \overrightarrow{\gamma},\,\delta_1 \qquad \delta_2 \text{ is fresh}}{\Psi\,;\Gamma,\ x : A \rightsquigarrow \Gamma,\ x :^{\delta_2} (A)^{\succeq \mathbf{!}} / \overrightarrow{\gamma},\,\delta_2}
$$

$$
\frac{\Psi \rightsquigarrow \Gamma / \overrightarrow{\gamma} \qquad \delta \text{ is fresh}}{\Psi\,;\varepsilon \rightsquigarrow \Gamma,\ \blacktriangleright^{\delta : \succeq \mathbf{!}} / \overrightarrow{\gamma},\,\delta}
\qquad
\frac{\Psi\,;\Delta_1\,;\cdots;\Delta_k \rightsquigarrow \Gamma / \overrightarrow{\gamma},\,\delta_0,\,\dots,\,\delta_k}{\Psi \rightsquigarrow \Gamma,\ \blacktriangleleft^{\delta_0} / \overrightarrow{\gamma},\,\delta_0}
$$

Figure 13 Translation from $\lambda^{\Box}$ to our lambda-calculus

▶ **Theorem 7.10** (On page 14). *If* $\varepsilon \vdash_{\mathbf{S4}} A$*, then* $\varepsilon \vdash (A)^{\succeq \mathbf{!}}$.

Proof. Direct result from Theorem E.6 ◀