Supplementary Material for “Refined^2 Environment Classifiers”

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List of Theorems

1 Outline

This is the supplementary material of the paper titled "Refined^2 Environment Classifiers".

2 Definitions

2.1 Surface Language

Integer	n	$\in \mathbf{Int}$		
Variable	x, y	$\in \mathbf{Var}$		
Expression	e^k	$\in \mathbf{Exp}^k$	$::=$	$n \mid x \mid \lambda x.\ e^k \mid {e_1}^k {e_2}^k \mid \langle e^{k+1} \rangle \mid \$_{k'}\{e^{k-k'}\}$ (where $k \geq k'$)
			$\mid$	$\mathbf{rec}\ f(x).\ e^k \mid \mathbf{ref}\,(e^k) \mid {e_1}^k := {e_2}^k \mid \mathbf{deref}(e^k)$

2.2 Surface Type System

2.2.1 Objects for the Type System

Classifier Stack	$\vec{\gamma}$	$::=$	$\varepsilon \mid \vec{\gamma}, \delta$
Polymorphic Classifier Declaration	ϱ	$::=$	$\varepsilon \mid \varrho, \gamma_1 :\succeq \gamma_2$
Base Type	ι	$::=$	$\mathtt{int} \mid \ldots$
Type	A, B	$::=$	$\iota \mid A \to B \mid \langle A \rangle^\gamma \mid A\ \mathbf{ref} \mid \forall \varrho. A$
Typing Context	Γ	$::=$	$\varepsilon \mid \Gamma, x{:}^\gamma A \mid \Gamma, \blacktriangleright^\gamma \mid \Gamma, \blacktriangleleft^\gamma_k \mid \Gamma, [\varrho]^\gamma$
Introduction Forms	${e_I}^k$	$::=$	$\lambda x.\ e^k \mid \mathbf{rec}\ f(x).\ e^k \mid \langle e^{k+1} \rangle$

Definition 1 (Free Occurrences of Classifiers)**.** $\mathbf{FC}(A)$, $\mathbf{FC}(\varrho)$, $\mathbf{FC}(\Gamma)$ *return the set of classifiers that occur freely in A, ϱ and Γ, respectively. They are defined as follows.*

$$\begin{aligned}
\mathbf{FC}(\iota) &= \emptyset \\
\mathbf{FC}(A_1 \to A_2) &= \mathbf{FC}(A_1) \cup \mathbf{FC}(A_2) \\
\mathbf{FC}(\langle A \rangle^\gamma) &= \mathbf{FC}(A) \cup \{\gamma\} \\
\mathbf{FC}(A\ \mathbf{ref}) &= \mathbf{FC}(A) \\
\mathbf{FC}(\forall \varrho. A) &= (\mathbf{FC}(A) - \mathbf{Dom}_C(\varrho)) \cup \mathbf{FC}(\varrho)
\end{aligned}$$

$$\begin{aligned}
\mathbf{FC}(\varepsilon) &= \emptyset \\
\mathbf{FC}(\varrho, \gamma_1 :\succeq \gamma_2) &= \mathbf{FC}(\varrho) \cup \{\gamma_2\}
\end{aligned}$$

$$\begin{aligned}
\mathbf{FC}(\varepsilon) &= \emptyset \\
\mathbf{FC}(x{:}^\gamma A, \Gamma) &= \mathbf{FC}(\Gamma) - \{\gamma\} \\
\mathbf{FC}(\blacktriangleright^\gamma, \Gamma) &= \mathbf{FC}(\Gamma) \cup \{\gamma\} \\
\mathbf{FC}(\blacktriangleleft^\gamma_k, \Gamma) &= \mathbf{FC}(\Gamma) - \{\gamma\} \\
\mathbf{FC}([\varrho]^\gamma, \Gamma) &= \mathbf{FC}(\Gamma) - (\{\gamma\} \cup \mathbf{Dom}_C(\varrho)) \cup \mathbf{FC}(\varrho)
\end{aligned}$$

Definition 2 (Classifier Domain of Contexts)**.** $\mathbf{Dom}_C(\Gamma)$ *and* $\mathbf{Dom}_C(\varrho)$ *return the set of classifiers declared in Γ and ϱ, respectively. They are defined as follows.*

$$\begin{aligned}
\mathbf{Dom}_C(\varepsilon) &= \emptyset \\
\mathbf{Dom}_C(\Gamma, x{:}^\gamma A) &= \mathbf{Dom}_C(\Gamma) \cup \{\gamma\} \\
\mathbf{Dom}_C(\Gamma, \blacktriangleright^\gamma) &= \mathbf{Dom}_C(\Gamma) \\
\mathbf{Dom}_C(\Gamma, \blacktriangleleft^\gamma_k) &= \mathbf{Dom}_C(\Gamma) \cup \{\gamma\} \\
\mathbf{Dom}_C(\Gamma, [\varrho]^\gamma) &= \mathbf{Dom}_C(\Gamma) \cup \mathbf{Dom}_C(\varrho) \cup \{\gamma\}
\end{aligned}$$

$$\begin{aligned}
\mathbf{Dom}_C(\varepsilon) &= \emptyset \\
\mathbf{Dom}_C(\varrho, \gamma_1 :\succeq \gamma_2) &= \mathbf{Dom}_C(\varrho) \cup \{\gamma_1\}
\end{aligned}$$

Definition 3 (Classifier Substitution Mapping)**.** *A classifier substitution mapping, ranged over by σ, is a finite mapping from classifiers to classifiers. We write it as $\gamma_1 \coloneqq \gamma'_1, \ldots, \gamma_k \coloneqq \gamma'_k$, assuming that the classifiers γ_1 to γ_k are pairwise distinct. Its domain and range are defined as follows:*

$$\begin{aligned}\mathbf{Dom}_C(\gamma_1 \coloneqq \gamma'_1, \ldots, \gamma_k \coloneqq \gamma'_k) &= \{\gamma_1, \ldots, \gamma_k\}\\ \mathbf{Reg}_C(\gamma_1 \coloneqq \gamma'_1, \ldots, \gamma_k \coloneqq \gamma'_k) &= \{\gamma'_1, \ldots, \gamma'_k\}\end{aligned}$$

Definition 4 (Classifier Substitution)**.** *Classifier substitution is the capture-avoiding operation induced by a classifier substitution mapping. For $\sigma = \gamma_1 \coloneqq \gamma'_1, \ldots, \gamma_k \coloneqq \gamma'_k$, we write $\delta[\sigma]$ for the classifier obtained by replacing δ with γ'_i when $\delta = \gamma_i$, and leaving δ unchanged otherwise.*

It extends to classifier sequences, types, polymorphic classifier declarations, and contexts as follows:

$$\begin{aligned}\varepsilon[\sigma] &= \varepsilon\\ (\overrightarrow{\gamma}, \delta)[\sigma] &= (\overrightarrow{\gamma})[\sigma], \delta[\sigma]\end{aligned}$$

$$\begin{aligned}\iota[\sigma] &= \iota\\ (A_1 \to A_2)[\sigma] &= A_1[\sigma] \to A_2[\sigma]\\ \langle A\rangle^{\gamma}[\sigma] &= \langle A[\sigma]\rangle^{\gamma[\sigma]}\\ (A\ \mathbf{ref})[\sigma] &= A[\sigma]\ \mathbf{ref}\\ (\forall\varrho.A)[\sigma] &= \forall\varrho[\sigma].A[\sigma]\\ &\qquad \textit{if } \mathbf{Dom}_C(\varrho) \cap (\mathbf{Dom}_C(\sigma) \cup \mathbf{Reg}_C(\sigma)) = \emptyset\end{aligned}$$

$$\begin{aligned}\varepsilon[\sigma] &= \varepsilon\\ (\varrho, \gamma_1 :\succeq \gamma_2)[\sigma] &= \varrho[\sigma], \gamma_1 :\succeq \gamma_2[\sigma]\end{aligned}$$

$$\begin{aligned}\varepsilon[\sigma] &= \varepsilon\\ (\Gamma, x{:}^{\gamma} A)[\sigma] &= \Gamma[\sigma], x{:}^{\gamma} A[\sigma]\\ (\Gamma, \blacktriangleright^{\gamma})[\sigma] &= \Gamma[\sigma], \blacktriangleright^{\gamma[\sigma]}\\ (\Gamma, \blacktriangleleft^{\gamma}_{k})[\sigma] &= \Gamma[\sigma], \blacktriangleleft^{\gamma}_{k}\\ (\Gamma, [\varrho]^{\gamma})[\sigma] &= \Gamma[\sigma], [\varrho[\sigma]]^{\gamma}\end{aligned}$$

The side condition in the clause for polymorphic classifier types ensures that the substitution does not capture the classifiers bound by ϱ.

2.2.2 Judgments and Derivation Rules

$\boxed{\Gamma \vdash^{\overrightarrow{\gamma}^{+}} \mathbf{wf}}$

$$\textsc{WF-Ctx-Emp}\ \frac{}{\varepsilon \vdash^{!} \mathbf{wf}} \qquad \textsc{WF-Ctx-Var}\ \frac{\Gamma \vdash^{\overrightarrow{\gamma_1},\gamma_2} \mathbf{wf} \qquad \Gamma \vdash^{\overrightarrow{\gamma_1},\gamma_2} A\ \mathbf{wf} \qquad \gamma_3 \notin \mathbf{Dom}_C(\Gamma)}{\Gamma, x{:}^{\gamma_3} A \vdash^{\overrightarrow{\gamma_1},\gamma_3} \mathbf{wf}}$$

$$\textsc{WF-Ctx-Open}\ \frac{\Gamma \vdash^{\overrightarrow{\gamma_1},\gamma_2} \mathbf{wf} \qquad \Gamma \vdash \gamma_3 \subseteq \gamma_2}{\Gamma, \blacktriangleright^{\gamma_3} \vdash^{\overrightarrow{\gamma_1},\gamma_2,\gamma_3} \mathbf{wf}} \qquad \textsc{WF-Ctx-Shut}\ \frac{\Gamma \vdash^{\overrightarrow{\gamma_1},\delta_0,\ldots,\delta_k} \mathbf{wf} \qquad \gamma_2 \notin \mathbf{Dom}_C(\Gamma)}{\Gamma, \blacktriangleleft^{\gamma_2}_{k} \vdash^{\overrightarrow{\gamma_1},\gamma_2} \mathbf{wf}}$$

$$\textsc{WF-Ctx-Polycls}\ \frac{\begin{array}{c}\Gamma \vdash^{\overrightarrow{\delta},\gamma'_0} \mathbf{wf} \qquad \gamma_0, \ldots, \gamma_k \text{ are pairwise distinct}\\ \gamma_i \notin \mathbf{Dom}_C(\Gamma) \text{ for } 0 \le i \le k \qquad \gamma_0 \notin \mathbf{Dom}_C((\gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k)) \qquad \Gamma \vdash \gamma'_{i+1} \subseteq \gamma'_i \text{ for } 0 \le i < k\end{array}}{\Gamma, [\gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k]^{\gamma_0} \vdash^{\overrightarrow{\delta},\gamma_0} \mathbf{wf}}$$

$\boxed{\Gamma \vdash^{\overrightarrow{\gamma}^+} A\ \mathbf{wf}}$

$$\textsc{WF-T-Base}\quad \frac{\Gamma \vdash^{\overrightarrow{\gamma}^+} \mathbf{wf}}{\Gamma \vdash^{\overrightarrow{\gamma}^+} \iota\ \mathbf{wf}} \qquad \textsc{WF-T-Func}\quad \frac{\Gamma \vdash^{\overrightarrow{\gamma}^+} A_1\ \mathbf{wf} \qquad \Gamma \vdash^{\overrightarrow{\gamma}^+} A_2\ \mathbf{wf}}{\Gamma \vdash^{\overrightarrow{\gamma}^+} A_1 \to A_2\ \mathbf{wf}}$$

$$\textsc{WF-T-}\langle\rangle\quad \frac{\Gamma, \blacktriangleright^{\gamma_1} \vdash^{\overrightarrow{\gamma_2}^+, \gamma_1} A\ \mathbf{wf}}{\Gamma \vdash^{\overrightarrow{\gamma_2}^+} \langle A \rangle^{\gamma_1}\ \mathbf{wf}} \qquad \textsc{WF-T-Ref}\quad \frac{\Gamma \vdash^{\overrightarrow{\gamma}^+} A\ \mathbf{wf}}{\Gamma \vdash^{\overrightarrow{\gamma}^+} A\ \mathbf{ref}\ \mathbf{wf}}$$

$$\textsc{WF-T-PolyCls}\quad \frac{\Gamma, [\varrho]^{\gamma_1} \vdash^{\overrightarrow{\delta}, \gamma_1} A\ \mathbf{wf} \qquad \mathbf{Dom}_C(\varrho) \cap \mathbf{FC}(A) = \emptyset \qquad \gamma_1 \notin \mathbf{FC}(A) \qquad \Gamma \vdash^{\overrightarrow{\delta}, \gamma_2} \mathbf{wf}}{\Gamma \vdash^{\overrightarrow{\delta}, \gamma_2} \forall \varrho. A\ \mathbf{wf}}$$

$\boxed{\Gamma \vdash \gamma_1 \preceq \gamma_2}$ $\boxed{\Gamma \vdash \gamma_1 \subseteq \gamma_2}$ Assuming that $\Gamma \vdash^{\overrightarrow{\delta}^+} \mathbf{wf}$ for some $\overrightarrow{\delta}^+$.

$$\preceq\textsc{-Refl}\quad \frac{\gamma \in \mathbf{Dom}_C(\Gamma) \cup \{\mathbf{!}\}}{\Gamma \vdash \gamma \preceq \gamma} \qquad \preceq\textsc{-Trans}\quad \frac{\Gamma \vdash \gamma_1 \preceq \gamma_2 \qquad \Gamma \vdash \gamma_2 \preceq \gamma_3}{\Gamma \vdash \gamma_1 \preceq \gamma_3} \qquad \preceq\textsc{-Var}\quad \frac{\Gamma_1 \vdash^{\overrightarrow{\gamma_1}, \gamma_2} \mathbf{wf}}{\Gamma_1, x{:}^{\gamma_3} A, \Gamma_2 \vdash \gamma_2 \preceq \gamma_3}$$

$$\preceq\textsc{-}\blacktriangleleft\quad \frac{\Gamma_1 \vdash^{\overrightarrow{\gamma_1}, \delta_0, \delta_1, \ldots, \delta_k} \mathbf{wf} \qquad k \geq 0}{\Gamma_1, \blacktriangleleft_k^{\gamma_2}, \Gamma_2 \vdash \delta_0 \preceq \gamma_2} \qquad \subseteq\textsc{-Lift}\quad \frac{\Gamma \vdash \gamma_1 \preceq \gamma_2}{\Gamma \vdash \gamma_1 \subseteq \gamma_2}$$

$$\subseteq\textsc{-Trans}\quad \frac{\Gamma \vdash \gamma_1 \subseteq \gamma_2 \qquad \Gamma \vdash \gamma_2 \subseteq \gamma_3}{\Gamma \vdash \gamma_1 \subseteq \gamma_3} \qquad \subseteq\textsc{-}\blacktriangleleft\quad \frac{\Gamma_1 \vdash^{\overrightarrow{\gamma_1}, \gamma_2} \mathbf{wf}}{\Gamma_1, \blacktriangleleft_k^{\gamma_3}, \Gamma_2 \vdash \gamma_2 \subseteq \gamma_3}$$

$$\preceq\textsc{-PolyCls}\quad \frac{\Gamma_1 \vdash^{\overrightarrow{\delta}, \gamma_0'} \mathbf{wf} \qquad 0 \leq i \leq k}{\Gamma_1, [\gamma_1 :\succeq \gamma_1', \ldots, \gamma_k :\succeq \gamma_k']^{\gamma_0}, \Gamma_2 \vdash \gamma_i' \preceq \gamma_i} \qquad \subseteq\textsc{-PolyCls}\quad \frac{0 \leq i < k}{\Gamma_1, [\gamma_1 :\succeq \gamma_1', \ldots, \gamma_k :\succeq \gamma_k']^{\gamma_0}, \Gamma_2 \vdash \gamma_{i+1} \subseteq \gamma_i}$$

$\boxed{\Gamma \vdash^{\overrightarrow{\gamma}^+} e : A}$

$$\textsc{T-Int}\quad \frac{\Gamma \vdash^{\overrightarrow{\gamma}^+} \mathbf{wf}}{\Gamma \vdash^{\overrightarrow{\gamma}^+} n : \texttt{int}} \qquad \textsc{T-Var}\quad \frac{\Gamma \vdash^{\overrightarrow{\gamma_1}, \gamma_2} \mathbf{wf} \qquad \mathbf{lookup}\,(\Gamma, x) = \mathbf{Done}\,(A, \gamma_3) \qquad \Gamma \vdash \gamma_3 \preceq \gamma_2}{\Gamma \vdash^{\overrightarrow{\gamma_1}, \gamma_2} x : A}$$

$$\textsc{T-Rec}\quad \frac{\Gamma, f{:}^{\gamma_1} A_1 \to A_2, x{:}^{\gamma_2} A_1 \vdash^{\overrightarrow{\gamma_3}, \gamma_2} e : A_2 \qquad \Gamma \vdash^{\overrightarrow{\gamma_3}, \gamma_4} \mathbf{wf}}{\Gamma \vdash^{\overrightarrow{\gamma_3}, \gamma_4} \mathbf{rec}\ f(x).\ e : A_1 \to A_2} \qquad \textsc{T-Func}\quad \frac{\Gamma, x{:}^{\gamma_1} A_1 \vdash^{\overrightarrow{\gamma_2}, \gamma_1} e : A_2 \qquad \gamma_1 \notin \mathbf{FC}(A_2) \qquad \Gamma \vdash^{\overrightarrow{\gamma_2}, \gamma_3} \mathbf{wf}}{\Gamma \vdash^{\overrightarrow{\gamma_2}, \gamma_3} \lambda x.\ e : A_1 \to A_2}$$

$$\textsc{T-App}\quad \frac{\Gamma \vdash^{\overrightarrow{\gamma}^+} e_1 : A_1 \to A_2 \qquad \Gamma \vdash^{\overrightarrow{\gamma}^+} e_2 : A_1}{\Gamma \vdash^{\overrightarrow{\gamma}^+} e_1\, e_2 : A_2}$$

$$\textsc{T-Quo}\quad \frac{\Gamma, \blacktriangleright^{\gamma_1} \vdash^{\overrightarrow{\gamma_2}^+, \gamma_1} e : A}{\Gamma \vdash^{\overrightarrow{\gamma_2}^+} \langle e \rangle : \langle A \rangle^{\gamma_1}} \qquad \textsc{T-Splice}\quad \frac{\Gamma \vdash^{\overrightarrow{\gamma_3}, \delta_0, \ldots, \delta_k} \mathbf{wf} \qquad \gamma_1 \notin \mathbf{FC}(A) \qquad \Gamma, \blacktriangleleft_k^{\gamma_1} \vdash^{\overrightarrow{\gamma_3}, \gamma_1} e : \langle A \rangle^{\gamma_2} \qquad \Gamma \vdash \gamma_2 \preceq \delta_k \qquad \gamma_1 \neq \gamma_2}{\Gamma \vdash^{\overrightarrow{\gamma_3}, \delta_0, \ldots, \delta_k} \$_k\{e\} : A}$$

$$\textsc{T-Ref}\quad \frac{\Gamma \vdash^{\overrightarrow{\gamma}^+} e : A}{\Gamma \vdash^{\overrightarrow{\gamma}^+} \mathbf{ref}\,(e) : A\ \mathbf{ref}} \qquad \textsc{T-Set}\quad \frac{\Gamma \vdash^{\overrightarrow{\gamma}^+} e_1 : A\ \mathbf{ref} \qquad \Gamma \vdash^{\overrightarrow{\gamma}^+} e_2 : A}{\Gamma \vdash^{\overrightarrow{\gamma}^+} e_1 := e_2 : A} \qquad \textsc{T-Get}\quad \frac{\Gamma \vdash^{\overrightarrow{\gamma}^+} e : A\ \mathbf{ref}}{\Gamma \vdash^{\overrightarrow{\gamma}^+} \mathbf{deref}(e) : A}$$

$$\textsc{T-PolyCls}\quad \frac{\Gamma, [\varrho]^{\gamma_1} \vdash^{\overrightarrow{\delta}, \gamma_1} e_I : A \qquad \mathbf{Dom}_C(\varrho) \cap \mathbf{FC}(A) = \emptyset \qquad \gamma_1 \notin \mathbf{FC}(A) \qquad \Gamma \vdash^{\overrightarrow{\delta}, \gamma_2} \mathbf{wf}}{\Gamma \vdash^{\overrightarrow{\delta}, \gamma_2} e_I : \forall \varrho. A}$$

T-AppCls

$$\frac{\Gamma \vdash^{\overrightarrow{\delta'}, \delta_0} e : \forall \gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k . A \qquad \Gamma \vdash \gamma'_i \preceq \delta_i \text{ for } 1 \le i \le k \qquad \Gamma \vdash \delta_{i+1} \subseteq \delta_i \text{ for } 0 \le i < k}{\Gamma \vdash^{\overrightarrow{\delta'}, \delta_0} e : A[\gamma_1 \coloneqq \delta_1, \ldots, \gamma_k \coloneqq \delta_k]}$$

Here, the auxiliary function $\textbf{lookup}\,(\Gamma, x)$ is defined as follows:

$$\begin{aligned}
\textbf{lookup}\,(\varepsilon, x) &= \textbf{Fail} \\
\textbf{lookup}\,((\Gamma, x{:}^{\gamma} A), x) &= \textbf{Done}\,(A, \gamma) \\
\textbf{lookup}\,((\Gamma, y{:}^{\gamma} A), x) &= \textbf{lookup}\,(\Gamma, x) \quad \text{if } x \ne y \\
\textbf{lookup}\,((\Gamma, \blacktriangleright^{\gamma}), x) &= \textbf{lookup}\,(\Gamma, x) \\
\textbf{lookup}\,((\Gamma, \blacktriangleleft^{\gamma}_{k}), x) &= \textbf{lookup}\,(\Gamma, x) \\
\textbf{lookup}\,((\Gamma, [\varrho]^{\gamma}), x) &= \textbf{lookup}\,(\Gamma, x)
\end{aligned}$$

2.3 Operational Semantics

2.3.1 Runtime Objects

Value	v	$\in$ **Val**	$::=$	$n \mid \textbf{clos}\,(E, R, \lambda x.\ e^0) \mid \textbf{clos}\,(E, R, \textbf{rec}\ f(x).\ e^0) \mid \langle e^0 \rangle \mid l$
Reserved Names	N	$\in$ **RN**	$::=$	$\varepsilon \mid N, x$
Value Environment	E	$\in$ **VEnv**	$::=$	$\varepsilon \mid E, x \coloneqq v$
Renaming Environment	R	$\in$ **REnv**	$::=$	$\textbf{id} \mid R, x \coloneqq y$
Store	S	$\in$ **Store**	$::=$	$\varepsilon \mid S, l \coloneqq v$

2.3.2 Notations for the Meta Language

For the following definitions, we briefly introduce definitions and notations for the meta language. The meta language is a simple functional language with pattern matching.

We have the type $\textbf{Result}(\tau)$ with the following definition:

$$\textbf{type Result}(\tau) \ ::= \textbf{Timeout} \mid \textbf{Fail} \mid \textbf{Done}(\tau)$$

Then, we introduce monadic notations for the type $\textbf{Result}(\tau)$ as follows:

$$\begin{aligned}
\textbf{return}(M) &\equiv \textbf{Done}(M) \\
X \leftarrow M_1;\ M_2 &\equiv \textbf{match}\ M_1\ \textbf{with} \\
&\ \textbf{case Timeout} \Rightarrow \textbf{Timeout} \\
&\ \textbf{case Fail} \Rightarrow \textbf{Fail} \\
&\ \textbf{case Done}(R) \Rightarrow [X \coloneqq R]M_2
\end{aligned}$$

2.3.3 Auxiliary Functions

$$\begin{aligned}
\textbf{lookup}\,(\varepsilon, x) &= \textbf{Fail} \\
\textbf{lookup}\,((E, x \coloneqq v), x) &= \textbf{Done}\,(v) \\
\textbf{lookup}\,((E, y \coloneqq v), x) &= \textbf{lookup}\,(E, x) \quad \text{if } x \ne y
\end{aligned}$$

$$\textbf{gensym}\,(N) = \textbf{return}\,(x) \quad \text{for some } x \text{ such that } x \notin N$$

$$\begin{aligned}
\textbf{rename}\,(\textbf{id}, x) &= x \\
\textbf{rename}\,((R, x \coloneqq y), x) &= y \\
\textbf{rename}\,((R, x \coloneqq y), z) &= \textbf{rename}\,(R, z) \quad \text{if } x \ne z
\end{aligned}$$

$$\mathbf{alloc}\,(S) = \mathbf{return}\,(l) \quad \text{for some } l \text{ such that } l \notin \mathbf{Dom}_L(S)$$

$$\begin{aligned}
\mathbf{lookupStore}(\varepsilon, l) &= \mathbf{Fail} \\
\mathbf{lookupStore}((S, l \coloneqq v), l) &= \mathbf{Done}\,(v) \\
\mathbf{lookupStore}((S, l_1 \coloneqq v), l_2) &= \mathbf{lookupStore}(S, l_2) \quad \text{if } l_1 \neq l_2
\end{aligned}$$

$$\begin{aligned}
\mathbf{updateStore}\,(\varepsilon, l, v) &= \mathbf{Fail} \\
\mathbf{updateStore}\,((S, l \coloneqq v_1), l, v_2) &= \mathbf{return}\,(S, l \coloneqq v_2) \\
\mathbf{updateStore}\,((S, l_1 \coloneqq v_1), l_2, v_2) &= S_1 \leftarrow \mathbf{updateStore}\,(S, l_2, v_2) \\
&\ \mathbf{return}\,(S_1, l_1 \coloneqq v_1) \quad \text{if } l_1 \neq l_2
\end{aligned}$$

$$\begin{aligned}
\mathbf{toQuote}\,(\langle e^0 \rangle) &= \mathbf{Done}\,(e^0) \\
\mathbf{toQuote}\,(v) &= \mathbf{Fail} \text{ otherwise}
\end{aligned}$$

$$\begin{aligned}
\mathbf{toLocation}\,(l) &= \mathbf{Done}\,(l) \\
\mathbf{toLocation}\,(v) &= \mathbf{Fail} \text{ otherwise}
\end{aligned}$$

2.3.4 Evaluator Functions

$$\begin{aligned}
\mathcal{V}^{\text{rt}}(e^0, E, R, N, S, 0) &= \mathbf{Timeout} \\
\mathcal{V}^{\text{rt}}(n, E, R, N, S, k^+) &= \mathbf{return}\,(n, N, S) \\
\mathcal{V}^{\text{rt}}(x, E, R, N, S, k^+) &= v \leftarrow \mathbf{lookup}\,(E, \mathbf{rename}\,(R, x)); \\
&\ \mathbf{return}\,(v, N, S) \\
\mathcal{V}^{\text{rt}}(\lambda x.\ e^0, E, R, N, S, k^+) &= \mathbf{return}\,(\mathbf{clos}\,(E, R, \lambda x.\ e^0), N, S) \\
\mathcal{V}^{\text{rt}}(\mathbf{rec}\ f(x).\ e^0, E, R, N, S, k^+) &= \mathbf{return}\,(\mathbf{clos}\,(E, R, \mathbf{rec}\ f(x).\ e^0), N, S) \\
\mathcal{V}^{\text{rt}}({e_1}^0\, {e_2}^0, E, R, N, S, k^+) &= (v_1, N_1, S_1) \leftarrow \mathcal{V}^{\text{rt}}({e_1}^0, E, R, N, S, k); \\
&\ (v_2, N_2, S_2) \leftarrow \mathcal{V}^{\text{rt}}({e_2}^0, E, R, N_1, S_1, k); \\
&\ \mathbf{match}\ v_1\ \mathbf{with} \\
&\ \mathbf{case\ clos}\,(E', R', \lambda x.\ {e_3}^0) \ \Rightarrow \\
&\ \quad y \leftarrow \mathbf{gensym}\,(N_2); \\
&\ \quad \mathcal{V}^{\text{rt}}({e_3}^0, (E', y \coloneqq v_2), (R', x \coloneqq y), (N_2, y), S_2, k) \\
&\ \mathbf{case\ clos}\,(E', R', \mathbf{rec}\ f(x).\ {e_3}^0) \ \Rightarrow \\
&\ \quad g \leftarrow \mathbf{gensym}\,(N_2); \\
&\ \quad y \leftarrow \mathbf{gensym}\,(N_2, g); \\
&\ \quad \mathcal{V}^{\text{rt}}({e_3}^0, (E', g \coloneqq v_1, y \coloneqq v_2), (R', f \coloneqq g, x \coloneqq y), (N_2, g, y), S_2, k) \\
&\ \mathbf{case}\ _ \Rightarrow \mathbf{Fail} \\
&\ \mathbf{end} \\
\mathcal{V}^{\text{rt}}(\langle {e_1}^1 \rangle, E, R, N, S, k^+) &= (v_1, N_1, S_1) \leftarrow \mathcal{V}^{\text{fut}}(0, {e_1}^1, E, R, N, S, k); \\
&\ \mathbf{return}\,(\langle v_1 \rangle, N_1, S_1) \\
\mathcal{V}^{\text{rt}}(\$_0\{{e_1}^0\}, E, R, N, S, k^+) &= (v_1, N_1, S_1) \leftarrow \mathcal{V}^{\text{rt}}({e_1}^0, E, R, N, S, k); \\
&\ {e_2}^0 \leftarrow \mathbf{toQuote}\,(v_1); \\
&\ \mathcal{V}^{\text{rt}}({e_2}^0, E, \mathbf{id}, N_1, S_1, k) \\
\mathcal{V}^{\text{rt}}(\mathbf{ref}\,(e^0), E, R, N, S, k^+) &= (v_1, N_1, S_1) \leftarrow \mathcal{V}^{\text{rt}}(e^0, E, R, N, S, k);
\end{aligned}$$

$$
\begin{aligned}
& l \leftarrow \textbf{alloc}\,(S_1); \\
& \textbf{return}\,(l, N_1, (S_1, l \coloneqq v_1)); \\
\mathcal{V}^{\mathrm{rt}}({e_1}^0 \coloneqq {e_2}^0, E, R, N, S, k^+) = {} & (v_1, N_1, S_1) \leftarrow \mathcal{V}^{\mathrm{rt}}({e_1}^0, E, R, N, S, k); \\
& (v_2, N_2, S_2) \leftarrow \mathcal{V}^{\mathrm{rt}}({e_2}^0, E, R, N_1, S_1, k); \\
& l \leftarrow \textbf{toLocation}\,(v_1); \\
& S_3 \leftarrow \textbf{updateStore}\,(S_2, l, v_2); \\
& \textbf{return}\,(v_2, N_2, S_3) \\
\mathcal{V}^{\mathrm{rt}}(\textbf{deref}(e^0), E, R, N, S, k^+) = {} & (v_1, N_1, S_1) \leftarrow \mathcal{V}^{\mathrm{rt}}(e^0, E, R, N, S, k); \\
& l \leftarrow \textbf{toLocation}\,(v_1); \\
& v_2 \leftarrow \textbf{lookupStore}(S_1, l); \\
& \textbf{return}\,(v_2, N_1, S_1)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}^{\mathrm{fut}}(k_1, e^{{k_1}^+}, E, R, N, S, 0) = {} & \textbf{Timeout} \\
\mathcal{V}^{\mathrm{fut}}(k_1, n, E, R, N, S, {k_2}^+) = {} & \textbf{return}\,(n, N, S) \\
\mathcal{V}^{\mathrm{fut}}(k_1, x, E, R, N, S, {k_2}^+) = {} & \textbf{return}\,(\textbf{rename}\,(R, x), N, S) \\
\mathcal{V}^{\mathrm{fut}}(k_1, \lambda \color{red}{x}\color{black}.\ {e_1}^{{k_1}^+}, E, R, N, S, {k_2}^+) = {} & y \leftarrow \textbf{gensym}\,(N); \\
& ({e_2}^{k_1}, N_1, S_1) \leftarrow \mathcal{V}^{\mathrm{fut}}(k_1, {e_1}^{{k_1}^+}, E, (R, x \coloneqq y), (N, y), S, k_2); \\
& \textbf{return}\,(\lambda \color{red}{y}\color{black}.\ {e_2}^{k_1}, N_1, S_1) \\
\mathcal{V}^{\mathrm{fut}}(k_1, \textbf{rec}\ \color{red}{f}\color{black}(\color{red}{x}\color{black}).\ {e_1}^{{k_1}^+}, E, R, N, S, {k_2}^+) = {} & g \leftarrow \textbf{gensym}\,(N); \\
& y \leftarrow \textbf{gensym}\,(N, g); \\
& ({e_2}^{k_1}, N_1, S_1) \leftarrow \mathcal{V}^{\mathrm{fut}}(k_1, {e_1}^{{k_1}^+}, E, (R, f \coloneqq g, x \coloneqq y), (N, g, y), S, k_2); \\
& \textbf{return}\,(\textbf{rec}\ \color{red}{g}\color{black}(\color{red}{y}\color{black}).\ {e_2}^{k_1}, N_1, S_1) \\
\mathcal{V}^{\mathrm{fut}}(k_1, {e_1}^{{k_1}^+}\, {e_2}^{{k_1}^+}, E, R, N, S, {k_2}^+) = {} & ({e_3}^{k_1}, N_1, S_1) \leftarrow \mathcal{V}^{\mathrm{fut}}(k_1, {e_1}^{{k_1}^+}, E, R, N, S, k_2); \\
& ({e_4}^{k_1}, N_2, S_2) \leftarrow \mathcal{V}^{\mathrm{fut}}(k_1, {e_2}^{{k_1}^+}, E, R, N_1, S_1, k_2); \\
& \textbf{return}\,({e_3}^{k_1}\, {e_4}^{k_1}, N_2, S_2) \\
\mathcal{V}^{\mathrm{fut}}(k_1, \langle {e_1}^{k_1+2} \rangle, E, R, N, S, {k_2}^+) = {} & ({e_2}^{{k_1}^+}, N_1, S_1) \leftarrow \mathcal{V}^{\mathrm{fut}}({k_1}^+, {e_1}^{k_1+2}, E, R, N, S, k_2); \\
& \textbf{return}\,(\langle {e_2}^{{k_1}^+} \rangle, N_1, S_1) \\
\mathcal{V}^{\mathrm{fut}}(k_1, \$_{k_1+}\{{e_1}^0\}, E, R, N, S, {k_2}^+) = {} & (v_1, N_1, S_1) \leftarrow \mathcal{V}^{\mathrm{rt}}({e_1}^0, E, R, N, S, k_2); \\
& {e_2}^0 \leftarrow \textbf{toQuote}\,(v_1); \\
& \textbf{return}\,({e_2}^0, N_1, S_1); \quad (\text{Note that } {e_2}^0 \in \textbf{Exp}^{k_1}) \\
\mathcal{V}^{\mathrm{fut}}(k_1, \$_{k_2}\{{e_1}^{{k_1}^+ - k_2}\}, E, R, N, S, {k_3}^+) \ & (\text{where } k_1 \geq k_2\ ) \\
= {} & ({e_2}^{k_1-k_2}, N_1, S_1) \leftarrow \mathcal{V}^{\mathrm{fut}}(k_1 - k_2, {e_1}^{{k_1}^+ - k_2}, E, R, N, S, k_3); \\
& \textbf{return}\,(\$_{k_2}\{{e_2}^{k_1-k_2}\}, N_1, S_1) \\
\mathcal{V}^{\mathrm{fut}}(k_1, \textbf{ref}\ ({e_1}^{{k_1}^+}), E, R, N, S, {k_2}^+) = {} & ({e_2}^{k_1}, N_1, S_1) \leftarrow \mathcal{V}^{\mathrm{fut}}(k_1, e^{{k_1}^+}, E, R, N, S, k_2); \\
& \textbf{return}\,(\textbf{ref}\ ({e_2}^{k_1}), N_1, S_1) \\
\mathcal{V}^{\mathrm{fut}}(k_1, {e_1}^{{k_1}^+} \coloneqq {e_2}^{{k_1}^+}, E, R, N, S, {k_2}^+) = {} & ({e_3}^{k_1}, N_1, S_1) \leftarrow \mathcal{V}^{\mathrm{fut}}(k_1, {e_1}^{{k_1}^+}, E, R, N, S, k_2); \\
& ({e_4}^{k_1}, N_2, S_2) \leftarrow \mathcal{V}^{\mathrm{fut}}(k_1, {e_2}^{{k_1}^+}, E, R, N_1, S_1, k_2); \\
& \textbf{return}\,({e_3}^{k_1} \coloneqq {e_4}^{k_1}, N_2, S_2) \\
\mathcal{V}^{\mathrm{fut}}(k_1, \textbf{deref}({e_1}^{{k_1}^+}), E, R, N, S, {k_2}^+) = {} & ({e_2}^{k_1}, N_1, S_1) \leftarrow \mathcal{V}^{\mathrm{fut}}(k_1, {e_1}^{{k_1}^+}, E, R, N, S, k_2); \\
& \textbf{return}\,(\textbf{deref}({e_2}^{k_1}), N_1, S_1)
\end{aligned}
$$

2.4 Runtime Typing

2.4.1 Objects for Runtime Typing

$$
\begin{array}{llll}
\textbf{Reserved Names Typing} & \Psi & ::= & \varepsilon \mid \Psi, x : A \mathbin{@} \gamma_1 :\succeq \gamma_2 \mid \Psi, \gamma_1(:\succeq \gamma_2 / :\supseteq \gamma_3) \mid \Psi, [\gamma_0 :\succeq \delta_0, \ldots, \gamma_k :\succeq \delta_k] \\
\textbf{Location Typing} & \Theta & ::= & \varepsilon \mid \Theta, l : A
\end{array}
$$

Definition 5. *1. We write $\Psi_1 \subseteq \Psi_2$ if there exists Ψ_3 such that $\Psi_1, \Psi_3 = \Psi_2$.*

2. We write $\Theta_1 \subseteq \Theta_2$ if there exists Θ_3 such that $\Theta_1, \Theta_3 = \Theta_2$.

Definition 6 (Classifier Domain of Reserved Names Typing)**.** $\mathbf{Dom}_C(\Psi)$ *returns the set of classifiers declared in* Ψ*, and is defined as follows.*

$$
\begin{aligned}
\mathbf{Dom}_C(\varepsilon) &= \{\mathbf{!}\} \\
\mathbf{Dom}_C(\Psi, \gamma_1(:\succeq \gamma_2 / :\supseteq \gamma_3)) &= \mathbf{Dom}_C(\Psi) \cup \{\gamma_1\} \\
\mathbf{Dom}_C(\Psi, x : A \mathbin{@} \gamma_1 :\succeq \gamma_2) &= \mathbf{Dom}_C(\Psi) \cup \{\gamma_1\} \\
\mathbf{Dom}_C(\Psi, [\gamma_0 :\succeq \delta_0, \ldots, \gamma_k :\succeq \delta_k]) &= \mathbf{Dom}_C(\Psi) \cup \{\gamma_0, \ldots, \gamma_k\}
\end{aligned}
$$

Definition 7 (Variable Domain of Reserved Names Typing)**.** $\mathbf{Dom}_V(\Psi)$ *returns the set of variables declared in* Ψ*, and is defined as follows.*

$$
\begin{aligned}
\mathbf{Dom}_V(\varepsilon) &= \emptyset \\
\mathbf{Dom}_V(\Psi, \gamma_1(:\succeq \gamma_2 / :\supseteq \gamma_3)) &= \mathbf{Dom}_V(\Psi) \\
\mathbf{Dom}_V(\Psi, x : A \mathbin{@} \gamma_1 :\succeq \gamma_2) &= \mathbf{Dom}_V(\Psi) \cup \{x\} \\
\mathbf{Dom}_V(\Psi, [\gamma_0 :\succeq \delta_0, \ldots, \gamma_k :\succeq \delta_k]) &= \mathbf{Dom}_V(\Psi)
\end{aligned}
$$

Definition 8 (Location Domain of Store Typing and Store)**.** $\mathbf{Dom}_L(\Theta)$ *and* $\mathbf{Dom}_L(S)$ *return the set of locations declared in* Θ *and* S*, respectively.*

$$
\begin{aligned}
\mathbf{Dom}_L(\varepsilon) &= \emptyset \\
\mathbf{Dom}_L(\Theta, l : A) &= \mathbf{Dom}_L(\Theta) \cup \{l\}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Dom}_L(\varepsilon) &= \emptyset \\
\mathbf{Dom}_L(S, l \coloneqq v) &= \mathbf{Dom}_L(S) \cup \{l\}
\end{aligned}
$$

2.4.2 Judgments and Derivation Rules

$\boxed{\vdash \Psi \ \mathbf{wf}}$

R-WF-NHT-Emp
$$\vdash \varepsilon \ \mathbf{wf}$$

R-WF-NHT-CLS
$$\frac{\vdash \Psi \ \mathbf{wf} \qquad \gamma_1 \notin \mathbf{Dom}_C(\Psi) \qquad \gamma_2 \in \mathbf{Dom}_C(\Psi) \qquad \gamma_3 \in \mathbf{Dom}_C(\Psi)}{\vdash \Psi, \gamma_1(:\succeq \gamma_2 / :\supseteq \gamma_3) \ \mathbf{wf}}$$

R-WF-NHT-Name
$$\frac{\vdash \Psi \ \mathbf{wf} \qquad x \notin \mathbf{Dom}_V(\Psi) \qquad \gamma_1 \in \mathbf{Dom}_C(\Psi) \qquad \gamma_2 \notin \mathbf{Dom}_C(\Psi) \qquad \Psi \vdash^{\gamma_1} A \ \mathbf{wf}}{\vdash \Psi, x : A \mathbin{@} \gamma_2 :\succeq \gamma_1 \ \mathbf{wf}}$$

R-WF-NHT-Polycls
$$\frac{\begin{array}{c}\vdash \Psi \ \mathbf{wf} \qquad \gamma_0, \ldots, \gamma_k \text{ are pairwise distinct} \\ \Psi \vdash \delta_{i+1} \subseteq \delta_i \text{ for } 0 \leq i < k \qquad \delta_i \in \mathbf{Dom}_C(\Psi) \text{ for } 0 \leq i \leq k \qquad \gamma_i \notin \mathbf{Dom}_C(\Psi) \text{ for } 0 \leq i \leq k\end{array}}{\vdash \Psi, [\gamma_0 :\succeq \delta_0, \ldots, \gamma_k :\succeq \delta_k] \ \mathbf{wf}}$$

$\boxed{\Psi \vdash \overrightarrow{\gamma}^{+} \ \mathbf{wf}}$

R-WF-CSeq-Init

$$\frac{\gamma \in \mathbf{Dom}_C(\Psi) \qquad \vdash \Psi\ \mathbf{wf}}{\Psi \vdash \gamma\ \mathbf{wf}}$$

R-WF-CSeq-Succ

$$\frac{\Psi \vdash \vec{\gamma_1}^{+}\ \mathbf{wf} \qquad \gamma_2 \in \mathbf{Dom}_C(\Psi)}{\Psi \vdash \vec{\gamma_1}^{+}, \gamma_2\ \mathbf{wf}}$$

$\boxed{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2}^{+}} \mathbf{wf}}$

R-WF-Ctx-Var

$$\frac{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2},\gamma_3} \mathbf{wf} \qquad \Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2},\gamma_3} A\ \mathbf{wf} \qquad \gamma_4 \notin \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma)}{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma, x{:}^{\gamma_4} A \vdash^{\vec{\gamma_2},\gamma_4} \mathbf{wf}}$$

R-WF-Ctx-Nil

$$\frac{\Psi \vdash \vec{\gamma}^{+}\ \mathbf{wf}}{\Psi \mid^{\vec{\gamma}^{+}} \varepsilon \vdash^{\vec{\gamma}^{+}} \mathbf{wf}}$$

R-WF-Ctx-▶

$$\frac{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2},\gamma_3} \mathbf{wf} \qquad \Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash \gamma_4 \subseteq \gamma_3}{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma, \blacktriangleright^{\gamma_4} \vdash^{\vec{\gamma_2},\gamma_3,\gamma_4} \mathbf{wf}}$$

R-WF-Ctx-◀

$$\frac{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2}^{+},\delta_0,\ldots,\delta_k} \mathbf{wf} \qquad \gamma_3 \notin \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma)}{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma, \blacktriangleleft^{\gamma_3}_{k} \vdash^{\vec{\gamma_2},\gamma_3} \mathbf{wf}}$$

R-WF-Ctx-PolyCls

$$\frac{\begin{array}{c}\Psi \mid^{\vec{\delta_1}^{+}} \Gamma \vdash^{\vec{\delta_2},\gamma_0'} \mathbf{wf} \qquad \gamma_0,\ldots,\gamma_k \text{ are pairwise distinct} \qquad \gamma_i \notin \mathbf{Dom}_C(\Psi) \text{ for } 0 \le i \le k \\ \gamma_i \notin \mathbf{Dom}_C(\Gamma) \text{ for } 0 \le i \le k \qquad \gamma_0 \notin \mathbf{Dom}_C((\gamma_1 :\succeq \gamma_1', \ldots, \gamma_k :\succeq \gamma_k')) \qquad \Psi \mid^{\vec{\delta_1}^{+}} \Gamma \vdash \gamma_{i+1}' \subseteq \gamma_i' \text{ for } 0 \le i < k\end{array}}{\Psi \mid^{\vec{\delta_1}^{+}} \Gamma, [\gamma_1 :\succeq \gamma_1', \ldots, \gamma_k :\succeq \gamma_k']^{\gamma_0} \vdash^{\vec{\delta_2},\gamma_0} \mathbf{wf}}$$

$\boxed{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2}^{+}} A\ \mathbf{wf}}$

- $\Psi \vdash^{\vec{\gamma}^{+}} A\ \mathbf{wf}$ is a shorthand for $\Psi \mid^{\vec{\gamma}^{+}} \varepsilon \vdash^{\vec{\gamma}^{+}} A\ \mathbf{wf}$.

R-WF-T-Base

$$\frac{\vdash \Psi\ \mathbf{wf} \qquad \Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2}^{+}} \mathbf{wf}}{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2}^{+}} \iota\ \mathbf{wf}}$$

R-WF-T-Func

$$\frac{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2}^{+}} A_1\ \mathbf{wf} \qquad \Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2}^{+}} A_2\ \mathbf{wf}}{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2}^{+}} A_1 \to A_2\ \mathbf{wf}}$$

R-WF-T-⟨⟩

$$\frac{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma, \blacktriangleright^{\gamma_2} \vdash^{\vec{\gamma_3}^{+},\gamma_2} A\ \mathbf{wf}}{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_3}^{+}} \langle A \rangle^{\gamma_2}\ \mathbf{wf}}$$

R-WF-T-Ref

$$\frac{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2}^{+}} A\ \mathbf{wf}}{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2}^{+}} A\ \mathbf{ref}\ \mathbf{wf}}$$

R-WF-T-PolyCls

$$\frac{\begin{array}{c}\Psi \mid^{\vec{\delta_1}^{+}} \Gamma, [\gamma_1 :\succeq \gamma_1', \ldots, \gamma_k :\succeq \gamma_k']^{\gamma_0} \vdash^{\vec{\delta_2},\gamma_0} A\ \mathbf{wf} \\ \gamma_i \notin \mathbf{FC}(A) \text{ for } 1 \le i \le k \qquad \gamma_0 \notin \mathbf{FC}(A) \qquad \Psi \mid^{\vec{\delta_1}^{+}} \Gamma \vdash^{\vec{\delta_1},\gamma_0'} \mathbf{wf}\end{array}}{\Psi \mid^{\vec{\delta_1}^{+}} \Gamma \vdash^{\vec{\delta_2},\gamma_0'} \forall \gamma_1 :\succeq \gamma_1', \ldots, \gamma_k :\succeq \gamma_k'.A\ \mathbf{wf}}$$

$\boxed{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash \gamma_2 \preceq \gamma_3}$ $\boxed{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash \gamma_2 \subseteq \gamma_3}$

- These judgments assume that $\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash^{\vec{\gamma_2}^{+}} \mathbf{wf}$ for some $\vec{\gamma_2}^{+}$.
- $\Psi \vdash \gamma_2 \preceq \gamma_3$ and $\Psi \vdash \gamma_2 \subseteq \gamma_3$ are shorthands for $\Psi \mid^{!} \varepsilon \vdash \gamma_2 \preceq \gamma_3$ and $\Psi \mid^{!} \varepsilon \vdash \gamma_2 \subseteq \gamma_3$, respectively.

R-≼-NHT-Name

$$\frac{x : A \mathbin{@} \gamma_1 :\succeq \gamma_2 \in \Psi}{\Psi \mid^{\vec{\gamma_3}^{+}} \Gamma \vdash \gamma_2 \preceq \gamma_1}$$

R-≼-NHT-◀

$$\frac{\gamma_1(:\succeq \gamma_2 / :\sqsupseteq \gamma_3) \in \Psi}{\Psi \mid^{\vec{\gamma_4}^{+}} \Gamma \vdash \gamma_2 \preceq \gamma_1}$$

R-≼-NHT-PolyCls

$$\frac{[\gamma_0 :\succeq \gamma_0', \ldots, \gamma_k :\succeq \gamma_k'] \in \Psi \qquad 0 \le i \le k}{\Psi \mid^{\vec{\delta}^{+}} \Gamma \vdash \gamma_i' \preceq \gamma_i}$$

R-≼-Var

$$\frac{\Psi \mid^{\vec{\gamma_1}} \Gamma_1 \vdash^{\vec{\gamma_2},\gamma_3} \mathbf{wf}}{\Psi \mid^{\vec{\gamma_1}} \Gamma_1, x{:}^{\gamma_4} A, \Gamma_2 \vdash \gamma_3 \preceq \gamma_4}$$

R-≼-◀-Ctx

$$\frac{\Psi \mid^{\vec{\gamma_1}} \Gamma_1 \vdash^{\vec{\gamma_2},\delta_0,\ldots,\delta_k} \mathbf{wf}}{\Psi \mid^{\vec{\gamma_1}} \Gamma_1, \blacktriangleleft^{\gamma_3}_{k}, \Gamma_2 \vdash \delta_0 \preceq \gamma_3}$$

R-≼-PolyCls-Ctx

$$\frac{\Psi \mid^{\vec{\delta}^{+}} \Gamma_1 \vdash^{\vec{\delta_1},\gamma_0'} \mathbf{wf} \qquad 0 \le i \le k}{\Psi \mid^{\vec{\delta}^{+}} \Gamma_1, [\gamma_1 :\succeq \gamma_1', \ldots, \gamma_k :\succeq \gamma_k']^{\gamma_0}, \Gamma_2 \vdash \gamma_i' \preceq \gamma_i}$$

R-≼-Refl

$$\frac{\gamma_1 \in \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma)}{\Psi \mid^{\vec{\gamma_2}} \Gamma \vdash \gamma_1 \preceq \gamma_1}$$

R-≼-Trans

$$\frac{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash \gamma_2 \preceq \gamma_3 \qquad \Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash \gamma_3 \preceq \gamma_4}{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash \gamma_2 \preceq \gamma_4}$$

R-⊆-Lift

$$\frac{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash \gamma_1 \preceq \gamma_2}{\Psi \mid^{\vec{\gamma_1}^{+}} \Gamma \vdash \gamma_1 \subseteq \gamma_2}$$

R-⊆-Trans
$$\frac{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash \gamma_2 \subseteq \gamma_3 \qquad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash \gamma_3 \subseteq \gamma_4}{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash \gamma_2 \subseteq \gamma_4}$$

R-⊆-NHT-◀
$$\frac{\gamma_1 (:\succeq \gamma_2 / :\supseteq \ \gamma_3) \in \Psi}{\Psi \mid^{\overrightarrow{\gamma_4}^+} \Gamma \vdash \gamma_3 \subseteq \gamma_1}$$

R-⊆-NHT-Polycls
$$\frac{[\gamma_0 :\succeq \gamma'_0, \ldots, \gamma_k :\succeq \gamma'_k] \in \Psi \qquad 0 \leq i < k}{\Psi \mid^{\overrightarrow{\delta}^+} \Gamma \vdash \gamma_{i+1} \subseteq \gamma_i}$$

R-⊆-◀-Ctx
$$\frac{\Psi \mid^{\overrightarrow{\gamma_1}} \Gamma_1 \vdash^{\overrightarrow{\gamma_2}, \delta_0, \ldots, \delta_k} \mathbf{wf}}{\Psi \mid^{\overrightarrow{\gamma_1}} \Gamma_1, \blacktriangleleft_k^{\gamma_3}, \Gamma_2 \vdash \delta_k \subseteq \gamma_3}$$

R-⊆-Polycls-Ctx
$$\frac{0 \leq i < k}{\Psi \mid^{\overrightarrow{\delta}^+} \Gamma_1, [\gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k]^{\gamma_0}, \Gamma_2 \vdash \gamma_{i+1} \subseteq \gamma_i}$$

$$\boxed{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} e : A}$$

R-T-Int
$$\frac{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_2}^+} \mathbf{wf}}{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} n : \mathtt{int}}$$

R-T-Var
$$\frac{\mathbf{lookup}\,(\Gamma, x) = \mathbf{Done}\,(A, \gamma_1) \qquad \Psi \mid^{\overrightarrow{\gamma_2}^+} \Gamma \vdash^{\overrightarrow{\gamma_3}, \gamma_4} \mathbf{wf} \qquad \Psi \mid^{\overrightarrow{\gamma_2}^+} \Gamma \vdash \gamma_1 \preceq \gamma_4}{\Psi \mid^{\overrightarrow{\gamma_2}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_3}, \gamma_4} x : A}$$

R-T-Name
$$\frac{\begin{array}{c}\mathbf{lookup}\,(\Gamma, x) = \mathbf{Fail} \\ y : A \mathbin{@} \gamma_1 :\succeq \gamma_2 \in \Psi \qquad \Psi \mid^{\overrightarrow{\gamma_3}^+} \Gamma \vdash^{\overrightarrow{\gamma_4}, \gamma_5} \mathbf{wf} \qquad \Psi \mid^{\overrightarrow{\gamma_3}^+} \Gamma \vdash \gamma_1 \preceq \gamma_5 \qquad \mathbf{rename}\,(R, x) = y\end{array}}{\Psi \mid^{\overrightarrow{\gamma_3}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_4}, \gamma_5} x : A}$$

R-T-Func
$$\frac{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma, x{:}^{\gamma_2} A_1 \vdash_R^{\overrightarrow{\gamma_3}, \gamma_2} e : A_2 \qquad \gamma_2 \notin \mathbf{FC}(A_2) \qquad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_3}, \gamma_4} \mathbf{wf}}{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_3}, \gamma_4} \lambda x.\ e : A_1 \to A_2}$$

R-T-Rec
$$\frac{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma, f{:}^{\gamma_2} A_1 \to A_2, x{:}^{\gamma_3} A_1 \vdash_R^{\overrightarrow{\gamma_4}, \gamma_3} e : A_2 \qquad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_4}, \gamma_5} \mathbf{wf}}{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_4}, \gamma_5} \mathbf{rec}\ f(x).\ e : A_1 \to A_2}$$

R-T-App
$$\frac{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} e_1 : A_1 \to A_2 \qquad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} e_2 : A_1}{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} e_1 e_2 : A_2}$$

R-T-Quote
$$\frac{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma, \blacktriangleright^{\gamma_2} \vdash_R^{\overrightarrow{\gamma_3}^+, \gamma_2} e_1 : A}{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_3}^+} \langle e_1 \rangle : \langle A \rangle^{\gamma_2}}$$

R-T-Splice
$$\frac{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_4}, \delta_0, \ldots, \delta_k} \mathbf{wf} \qquad \gamma_2 \notin \mathbf{FC}(A) \qquad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma, \blacktriangleleft_k^{\gamma_2} \vdash_R^{\overrightarrow{\gamma_4}, \gamma_2} e : \langle A \rangle^{\gamma_3} \qquad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash \gamma_3 \preceq \delta_k \qquad \gamma_2 \neq \gamma_3}{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_4}, \delta_0, \ldots, \delta_k} \$_k\{e\} : A}$$

R-T-Ref
$$\frac{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} e : A}{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} \mathbf{ref}\,(e) : A\ \mathbf{ref}}$$

R-T-Deref
$$\frac{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} e : A\ \mathbf{ref}}{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} \mathbf{deref}(e) : A}$$

R-T-Assign
$$\frac{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} e_1 : A\ \mathbf{ref} \qquad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} e_2 : A}{\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} e_1 \coloneqq e_2 : A}$$

R-T-Polycls
$$\frac{\begin{array}{c}\Psi \mid^{\overrightarrow{\delta_1}^+} \Gamma, [\gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k]^{\gamma_0} \vdash_R^{\overrightarrow{\delta_2}, \gamma_0} e_I : A \\ \gamma_i \notin \mathbf{FC}(A) \text{ for } 1 \leq i \leq k \qquad \gamma_0 \notin \mathbf{FC}(A) \qquad \Psi \mid^{\overrightarrow{\delta_1}^+} \Gamma \vdash^{\overrightarrow{\delta_2}, \gamma'_0} \mathbf{wf}\end{array}}{\Psi \mid^{\overrightarrow{\delta_1}^+} \Gamma \vdash_R^{\overrightarrow{\delta_2}, \gamma'_0} e_I : \forall \gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k.A}$$

R-T-Appcls
$$\frac{\begin{array}{c}\Psi \mid^{\overrightarrow{\delta'_1}^+} \Gamma \vdash_R^{\overrightarrow{\delta'_2}, \delta_0} e : \forall \gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k.A \\ \Psi \mid^{\overrightarrow{\delta'_1}^+} \Gamma \vdash \gamma'_i \preceq \delta_i \text{ for } 1 \leq i \leq k \qquad \Psi \mid^{\overrightarrow{\delta'_1}^+} \Gamma \vdash \delta_{i+1} \subseteq \delta_i \text{ for } 0 \leq i < k\end{array}}{\Psi \mid^{\overrightarrow{\delta'_1}^+} \Gamma \vdash_R^{\overrightarrow{\delta'_2}, \delta_0} e : A[\gamma_1 \coloneqq \delta_1, \ldots, \gamma_k \coloneqq \delta_k]}$$

$\boxed{\Psi \vdash \Theta\ \mathbf{wf}}$

$$\frac{}{\Psi \vdash \varepsilon\ \mathbf{wf}}\ \textsc{WF-St-Nil}$$

$$\frac{\Psi \vdash \Theta\ \mathbf{wf} \qquad l \notin \mathbf{Dom}_L(\Theta) \qquad \Psi \vdash^{\gamma} A\ \mathbf{wf} \text{ for some } \gamma}{\Psi \vdash (\Theta, l : A)\ \mathbf{wf}}\ \textsc{WF-St-Cons}$$

$\boxed{\Psi \vdash S \colon \Theta}$

$$\frac{\begin{array}{c} S = (l_1 \coloneqq v_1, \ldots, l_k \coloneqq v_k) \\ \Theta = (l_1 : A_1, \ldots, l_k : A_k) \qquad \Psi \vdash \Theta\ \mathbf{wf} \qquad \Psi \mid \Theta \vdash^{\gamma_i} v_i : A_i\ \mathbf{val} \text{ with some } \gamma_i \text{ for } 1 \le i \le k \end{array}}{\Psi \vdash S \colon \Theta}\ \textsc{T-Store}$$

$\boxed{\Psi \mid \Theta \vdash^{\gamma} v : A\ \mathbf{val}}$

$$\frac{\Psi \vdash \Theta\ \mathbf{wf} \qquad \Psi \vdash \gamma\ \mathbf{wf}}{\Psi \mid \Theta \vdash^{\gamma} n : \mathtt{int}\ \mathbf{val}}\ \textsc{VT-Int}$$

$$\frac{\Psi \mid^{\gamma_1} \varepsilon \vdash^{\gamma_1}_{R} \lambda x.\ e^0 : A \qquad \Psi \mid \Theta \vdash^{\gamma_1} E\ \mathbf{wf} \qquad \Psi \vdash^{\delta} A\ \mathbf{wf}}{\Psi \mid \Theta \vdash^{\delta} \mathbf{clos}\,(E, R, \lambda x.\ e^0) : A\ \mathbf{val}}\ \textsc{VT-Clos-Func}$$

$$\frac{\Psi \mid^{\gamma_1} \varepsilon \vdash^{\gamma_1}_{R} \mathbf{rec}\ f(x).\ e^0 : A \qquad \Psi \mid \Theta \vdash^{\gamma_1} E\ \mathbf{wf} \qquad \Psi \vdash^{\delta} A\ \mathbf{wf}}{\Psi \mid \Theta \vdash^{\delta} \mathbf{clos}\,(E, R, \mathbf{rec}\ f(x).\ e^0) : A\ \mathbf{val}}\ \textsc{VT-Clos-Rec}$$

$$\frac{\Psi \mid^{\gamma_1} \varepsilon \vdash^{\gamma_1}_{\mathbf{id}} \langle e^0 \rangle : A \qquad \Psi \vdash \Theta\ \mathbf{wf}}{\Psi \mid \Theta \vdash^{\gamma_1} \langle e^0 \rangle : A\ \mathbf{val}}\ \textsc{VT-Code}$$

$$\frac{\Psi \vdash \Theta\ \mathbf{wf} \qquad \mathbf{lookupStore}(\Theta, l) = \mathbf{Done}\,(A) \qquad \Psi \vdash^{\gamma} A\ \mathbf{wf}}{\Psi \mid \Theta \vdash^{\gamma} l : A\ \mathbf{ref}\ \mathbf{val}}\ \textsc{VT-Loc}$$

$\boxed{\Psi \mid \Theta \vdash^{\gamma} E\ \mathbf{wf}}$

$$\frac{\Psi \vdash \Theta\ \mathbf{wf}}{\Psi \mid \Theta \vdash^{!} \varepsilon\ \mathbf{wf}}\ \textsc{WF-VEnv-Empty}$$

$$\frac{\Psi \mid \Theta \vdash^{\gamma_1} E\ \mathbf{wf} \qquad \gamma_2 (:\succeq \gamma_1 / :\supseteq\ \gamma_3) \in \Psi}{\Psi \mid \Theta \vdash^{\gamma_2} E\ \mathbf{wf}}\ \textsc{WF-VEnv-Shut}$$

$$\frac{\Psi \mid \Theta \vdash^{\gamma_1} E\ \mathbf{wf} \qquad \Psi \mid \Theta \vdash^{\gamma_1} v : A\ \mathbf{val} \qquad x : A \mathbin{@} \gamma_2 :\succeq \gamma_1 \in \Psi}{\Psi \mid \Theta \vdash^{\gamma_2} E, x \coloneqq v\ \mathbf{wf}}\ \textsc{WF-VEnv-Var}$$

$$\frac{\Psi \mid \Theta \vdash^{\delta_0} E\ \mathbf{wf} \qquad [\gamma_0 :\succeq \delta_0, \ldots, \gamma_k :\succeq \delta_k] \in \Psi}{\Psi \mid \Theta \vdash^{\gamma_0} E\ \mathbf{wf}}\ \textsc{WF-VEnv-Polycls}$$

Here, $\mathbf{lookupStore}(\Theta, l)$ returns the type of location l in Θ if it exists, and **Fail** otherwise.

$$\begin{aligned} \mathbf{lookupStore}(\varepsilon, l) &= \mathbf{Fail} \\ \mathbf{lookupStore}(\Theta, l) &= \mathbf{Done}\,(A) \text{ if } \Theta = (\Theta', l : A) \\ \mathbf{lookupStore}(\Theta, l) &= \mathbf{lookupStore}(\Theta', l) \text{ if } \Theta = (\Theta', l' : A') \text{ and } l' \neq l \end{aligned}$$

3 Proofs

3.1 Basic Properties of Judgments

Lemma 1 (Regularity of Runtime Judgments)**.** *1. If* $\vdash \Psi_1, \Psi_2\ \mathbf{wf}$*, then* $\vdash \Psi_1\ \mathbf{wf}$*.*

2. If $\vdash \Psi_1, x : A \mathbin{@} \gamma_1 :\succeq \gamma_2, \Psi_2\ \mathbf{wf}$*, then* $\Psi_1 \vdash^{\gamma_2} A\ \mathbf{wf}$*.*

We also use the following entry-wise consequences of reserved names typing well-formedness:

(a) *If* $\vdash \Psi_1, x : A \mathbin{@} \gamma_2 :\succeq \gamma_1, \Psi_2$ **wf**, *then* $\gamma_1 \in \mathbf{Dom}_C(\Psi_1)$.

(b) *If* $\vdash \Psi_1, \gamma_1(:\succeq \gamma_2/ :\supseteq \gamma_3), \Psi_2$ **wf**, *then* $\gamma_2 \in \mathbf{Dom}_C(\Psi_1)$ *and* $\gamma_3 \in \mathbf{Dom}_C(\Psi_1)$.

(c) *If* $\vdash \Psi_1, [\gamma_0 :\succeq \delta_0, \ldots, \gamma_k :\succeq \delta_k], \Psi_2$ **wf**, *then* $\delta_i \in \mathbf{Dom}_C(\Psi_1)$ *for any* $0 \le i \le k$.

3. *If* $\Psi \vdash \overrightarrow{\gamma_1}^+ \mathbf{wf}$, *then* $\gamma_2 \in \mathbf{Dom}_C(\Psi)$ *for any* $\gamma_2 \in \{\overrightarrow{\gamma_1}^+\}$.
4. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash^{\overrightarrow{\gamma_2}^+} \mathbf{wf}$, *then* $\vdash \Psi$ **wf** *and* $\Psi \vdash \overrightarrow{\gamma_1}^+ \mathbf{wf}$.
5. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2 \vdash^{\overrightarrow{\gamma_2}^+} \mathbf{wf}$, *then* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash^{\overrightarrow{\gamma_3}^+} \mathbf{wf}$ *for some* $\overrightarrow{\gamma_3}^+$.
6. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash^{\overrightarrow{\gamma_2}^+, \overrightarrow{\gamma_3}} \mathbf{wf}$ *and* $\Gamma_1 = \Gamma_2, \Gamma_3$, *then* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_2 \vdash^{\overrightarrow{\gamma_2}^+} \mathbf{wf}$.
7. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, x{:}^{\gamma_1} A, \Gamma_3 \vdash^{\overrightarrow{\gamma_2}^+} \mathbf{wf}$, *then* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash^{\overrightarrow{\gamma_3}^+} A\ \mathbf{wf}$ *for some* $\overrightarrow{\gamma_3}^+$.
8. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_2}^+} \mathbf{wf}$ *and* $\delta \in \{\overrightarrow{\gamma_2}^+\}$, *then* $\delta \in \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma)$.
9. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_2}^+} \mathbf{wf}$, *then* $\mathbf{FC}(\Gamma) \subseteq \mathbf{Dom}_C(\Psi)$.
10. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_2}^+} A\ \mathbf{wf}$, *then* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_2}^+} \mathbf{wf}$.
11. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash \gamma_2 \preceq \gamma_3$ *or* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash \gamma_2 \subseteq \gamma_3$, *then* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_4}^+} \mathbf{wf}$ *for some* $\overrightarrow{\gamma_4}^+$.
12. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash \gamma_2 \preceq \gamma_3$ *or* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash \gamma_2 \subseteq \gamma_3$, *then* $\gamma_2 \in \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma)$ *and* $\gamma_3 \in \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma)$.
13. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} e : A$, *then* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_2}^+} A\ \mathbf{wf}$.
14. *If* $\Psi \vdash \Theta$ **wf**, *then* $\vdash \Psi$ **wf**.
15. *If* $\Psi \vdash \Theta_1, \Theta_2$ **wf**, *then* $\Psi \vdash \Theta_1$ **wf**.
16. *If* $\Psi \vdash \Theta_1, l : A, \Theta_2$ **wf**, *then* $\Psi \vdash^{\gamma} A$ **wf** *for some* γ.
17. *If* $\Psi \vdash S{:}\ \Theta$, *then* $\Psi \vdash \Theta$ **wf**.
18. *If* $\Psi \mid \Theta \vdash^{\gamma} v : A$ **val**, *then* $\Psi \vdash^{\gamma} A$ **wf** *and* $\Psi \vdash \Theta$ **wf**.
19. *If* $\Psi \mid \Theta \vdash^{\gamma} E$ **wf**, *then* $\Psi \vdash \gamma$ **wf** *and* $\Psi \vdash \Theta$ **wf**.
20. *If* $\Psi \mid \Theta \vdash^{\gamma_1} E_1, E_2$ **wf**, *then* $\Psi \mid \Theta \vdash^{\gamma_2} E_1$ **wf** *for some* γ_2.
21. *If* $\Psi \mid \Theta \vdash^{\gamma_1} E_1, x \coloneqq v, E_2$ **wf**, *then* $x \in \mathbf{Dom}_V(\Psi)$ *and* $\Psi \mid \Theta \vdash^{\gamma_2} v : A$ **val** *for some* A *and* γ_2.

Proof. By induction on the derivation of each judgment. The entry-wise consequences for reserved names typing well-formedness follow by inversion of the corresponding reserved names typing well-formedness rule, after peeling any suffix of the reserved names typing with the first item. The classifier-stack containment fact and the reachability/visibility endpoint fact are proved simultaneously: the context-variable and context-shut cases use classifier-stack containment for the relevant prefix, while the open-context case uses the visibility endpoint fact. The reserved names typing cases of reachability and visibility use the entry-wise consequences above. For Item 13, we use Lemma 2. ☐

Definition 9 (Stratified Contexts)**.** *For any* k_1, k_2 *such that* $0 \le k_1 \le k_2$, $\mathbf{Ctx}^{k_1,k_2}$ *is the set of contexts. We annotate a context like* Γ^{k_1,k_2} *to clarify that it is a member of* $\mathbf{Ctx}^{k_1,k_2}$. *They are defined mutually recursively as follows.*

$$\Gamma^{k_1,k_2} \in \mathbf{Ctx}^{k_1,k_2} ::= \varepsilon (\textit{where } k_1 = k_2\ ) \mid x{:}^{\gamma} A, \Gamma^{k_1,k_2} \mid \blacktriangleright^{\delta}, \Gamma^{k_1+1,k_2} \mid \blacktriangleleft^{\gamma}_{k_3}, \Gamma^{k_1-k_3,k_2} (\textit{where } k_1 \ge k_3\ ) \mid [\varrho]^{\gamma}, \Gamma^{k_1,k_2}$$

Lemma 2 (Monotonicity for Runtime Reachability, Visibility, and Type Well-formedness)**.** *1. If* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_2^{0,k} \vdash \delta_1 \preceq \delta_2$, $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1 \vdash^{\cdots,\gamma_4} \mathbf{wf}$, $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_3 \vdash^{\cdots,\gamma_5} \mathbf{wf}$, $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_3, \Gamma_2^{0,k} \vdash^{\cdots} \mathbf{wf}$, *and* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_3 \vdash \gamma_4 \preceq \gamma_5$, *then* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_3, \Gamma_2^{0,k} \vdash \delta_1 \preceq \delta_2$.

2. *If* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_2^{0,k} \vdash \delta_1 \subseteq \delta_2$, $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1 \vdash^{\dots,\gamma_4} \mathbf{wf}$, $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_3 \vdash^{\dots,\gamma_5} \mathbf{wf}$, $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_3, \Gamma_2^{0,k} \vdash^{\cdots} \mathbf{wf}$, *and* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_3 \vdash \gamma_4 \preceq \gamma_5$, *then* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_3, \Gamma_2^{0,k} \vdash \delta_1 \subseteq \delta_2$.

3. *If* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_2^{0,k} \vdash^{\overrightarrow{\gamma_3}^+} A\ \mathbf{wf}$, $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1 \vdash^{\dots,\gamma_4} \mathbf{wf}$, $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_3 \vdash^{\dots,\gamma_5} \mathbf{wf}$, $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_3, \Gamma_2^{0,k} \vdash^{\cdots} \mathbf{wf}$, *and* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_3 \vdash \gamma_4 \preceq \gamma_5$, *then* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1, \Gamma_3, \Gamma_2^{0,k} \vdash^{\overrightarrow{\gamma_6}^+} A\ \mathbf{wf}$ *for some* $\overrightarrow{\gamma_6}^+$.

Proof. By induction on the derivation of each judgment. $\square$

3.2 Properties on Reachability/Visibility Judgments

Definition 10. *Given* $\Psi \mid^{\delta_1} \Gamma \vdash^{\overrightarrow{\delta_2}} \mathbf{wf}$ *and* $\gamma \in \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma)$, $\mathbf{len}(\Psi, \Gamma)$ *returns the number of classifiers declared in* Ψ *and* Γ.

$$
\begin{aligned}
\mathbf{len}(\varepsilon, \varepsilon) &= 0 \\
\mathbf{len}((\Psi, x : A \mathbin{@} \gamma_1 :\succeq \gamma_2), \varepsilon) &= \mathbf{len}(\Psi, \varepsilon) + 1 \\
\mathbf{len}((\Psi, \gamma_1(:\succeq \gamma_2 / :\supseteq\ \gamma_3)), \varepsilon) &= \mathbf{len}(\Psi, \varepsilon) + 1 \\
\mathbf{len}((\Psi, [\gamma_1 :\succeq \delta_1, \dots, \gamma_k :\succeq \delta_k]), \varepsilon) &= \mathbf{len}(\Psi, \varepsilon) + k \\
\mathbf{len}(\Psi, (\Gamma, x{:}^{\gamma} A)) &= \mathbf{len}(\Psi, \Gamma) + 1 \\
\mathbf{len}(\Psi, (\Gamma, \blacktriangleright^{\gamma})) &= \mathbf{len}(\Psi, \Gamma) \\
\mathbf{len}(\Psi, (\Gamma, \blacktriangleleft^{\gamma}_{k})) &= \mathbf{len}(\Psi, \Gamma) + 1 \\
\mathbf{len}(\Psi, (\Gamma, [\gamma_1 :\succeq \gamma_1', \dots, \gamma_k :\succeq \gamma_k']^{\gamma_0})) &= \mathbf{len}(\Psi, \Gamma) + k + 1
\end{aligned}
$$

We also define $\mathbf{deg}(\Psi, \Gamma, \gamma)$ *as the position at which* γ *occurs among the classifiers declared in* Ψ *and* Γ.

$$
\begin{aligned}
\mathbf{deg}(\Psi, \Gamma, \mathbf{!}) &= 0 \\
\mathbf{deg}((\Psi_1, x : A \mathbin{@} \gamma_1 :\succeq \gamma_2, \Psi_2), \Gamma, \gamma_1) &= \mathbf{len}(\Psi_1, \varepsilon) + 1 \\
\mathbf{deg}((\Psi_1, \gamma_1(:\succeq \gamma_2 / :\supseteq\ \gamma_3), \Psi_2), \Gamma, \gamma_1) &= \mathbf{len}(\Psi_1, \varepsilon) + 1 \\
\mathbf{deg}((\Psi_1, [\gamma_1 :\succeq \delta_1, \dots, \gamma_k :\succeq \delta_k], \Psi_2), \Gamma, \gamma_i) &= \mathbf{len}(\Psi_1, \varepsilon) + k - i + 1 \\
\mathbf{deg}(\Psi, (\Gamma_1, x{:}^{\gamma} A, \Gamma_2), \gamma) &= \mathbf{len}(\Psi, \Gamma_1) + 1 \\
\mathbf{deg}(\Psi, (\Gamma_1, \blacktriangleleft^{\gamma}_{k}, \Gamma_2), \gamma) &= \mathbf{len}(\Psi, \Gamma_1) + 1 \\
\mathbf{deg}(\Psi, (\Gamma_1, [\gamma_1 :\succeq \gamma_1', \dots, \gamma_k :\succeq \gamma_k']^{\gamma_0}, \Gamma_2), \gamma_i) &= \mathbf{len}(\Psi, \Gamma_1) + k - i + 1
\end{aligned}
$$

Lemma 3. *If either* $\Psi \mid^{\gamma_1} \Gamma \vdash \gamma_2 \preceq \gamma_3$ *or* $\Psi \mid^{\gamma_1} \Gamma \vdash \gamma_2 \subseteq \gamma_3$ *holds, then* $\mathbf{deg}(\Psi, \Gamma, \gamma_2) \leq \mathbf{deg}(\Psi, \Gamma, \gamma_3)$. *Moreover, the inequality is strict whenever* $\gamma_2 \neq \gamma_3$.

Proof. By induction on the derivation of $\Psi \mid^{\gamma_1} \Gamma_2 \vdash \gamma_2 \preceq \gamma_3$ or $\Psi \mid^{\gamma_1} \Gamma_2 \vdash \gamma_2 \subseteq \gamma_3$. $\square$

Lemma 4 (Antisymmetry of $\preceq$ and $\subseteq$)**.** *If* $\Psi \mid^{\gamma_1} \Gamma \vdash \gamma_2 \preceq \gamma_3$ *and* $\Psi \mid^{\gamma_1} \Gamma \vdash \gamma_3 \preceq \gamma_2$, *then* $\gamma_2 = \gamma_3$. *Similarly, If* $\Psi \mid^{\gamma_1} \Gamma \vdash \gamma_2 \subseteq \gamma_3$ *and* $\Psi \mid^{\gamma_1} \Gamma \vdash \gamma_3 \subseteq \gamma_2$, *then* $\gamma_2 = \gamma_3$.

Proof. We show the first statement; the second is analogous. Suppose, for contradiction, that $\gamma_2 \neq \gamma_3$. By Lemma 3 applied to $\Psi \mid^{\gamma_1} \Gamma \vdash \gamma_2 \preceq \gamma_3$, we have $\mathbf{deg}(\Psi, \Gamma, \gamma_2) < \mathbf{deg}(\Psi, \Gamma, \gamma_3)$. Applying the same lemma to $\Psi \mid^{\gamma_1} \Gamma \vdash \gamma_3 \preceq \gamma_2$ gives $\mathbf{deg}(\Psi, \Gamma, \gamma_3) < \mathbf{deg}(\Psi, \Gamma, \gamma_2)$. This is impossible. Hence $\gamma_2 = \gamma_3$. $\square$

Lemma 5. 1. *If* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \preceq \gamma_2$ *and* $\gamma_2 \in \mathbf{Dom}_C(\Psi)$, *then* $\Psi \vdash \gamma_1 \preceq \gamma_2$.

2. *If* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \subseteq \gamma_2$ *and* $\gamma_2 \in \mathbf{Dom}_C(\Psi)$, *then* $\Psi \vdash \gamma_1 \subseteq \gamma_2$.

Proof. We prove both statements by induction on the given derivation. Since $\gamma_2 \in \mathbf{Dom}_C(\Psi)$, Lemma 3 implies $\gamma_1 \in \mathbf{Dom}_C(\Psi)$; the remaining argument is straightforward. $\square$

Lemma 6. 1. *If* $\Psi \vdash \gamma_1 \preceq \gamma_2$ *and* $\Psi \mid^{\delta} \Gamma \vdash^{\cdots} \mathbf{wf}$, *then* $\Psi \mid^{\delta} \Gamma \vdash \gamma_1 \preceq \gamma_2$.

2. *If* $\Psi \vdash \gamma_1 \subseteq \gamma_2$ *and* $\Psi \mid^{\delta} \Gamma \vdash^{\cdots} \mathbf{wf}$, *then* $\Psi \mid^{\delta} \Gamma \vdash \gamma_1 \subseteq \gamma_2$.

Proof. By induction on the given derivation. $\square$

Lemma 7 (Inversion for $\preceq$ w.r.t. Reserved Names Typing)**.** *If* $\Psi \vdash \gamma_1 \preceq \gamma_2$ *and* $\gamma_1 \neq \gamma_2$*, then the following hold:*

1. *If* $x : A \mathbin{@} \gamma_2 :\succeq \delta_1 \in \Psi$*, then* $\Psi \vdash \gamma_1 \preceq \delta_1$*.*
2. *If* $\gamma_2(:\succeq \delta_1 / :\supseteq \delta_2) \in \Psi$*, then* $\Psi \vdash \gamma_1 \preceq \delta_1$*.*
3. *If* $[\gamma_0 :\succeq \gamma'_0, \ldots, \gamma_k :\succeq \gamma'_k] \in \Psi$ *and* $\gamma_2 = \gamma_i$ *for some* i *with* $0 \leq i \leq k$*, then* $\Psi \vdash \gamma_1 \preceq \gamma'_i$*.*

Proof. By induction on the given derivation. $\square$

Lemma 8 (Inversion for $\preceq$ w.r.t. Context)**.** *If* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \preceq \gamma_2$*,* $\gamma_1 \neq \gamma_2$ *and* $\gamma_2 \in \mathbf{Dom}_C(\Gamma)$*, then the following hold:*

1. *If* $\Gamma = \Gamma_1, x{:}^{\gamma_2} A, \Gamma_2$ *and* $\Psi \mid^{\delta_1} \Gamma_1 \vdash^{\ldots,\delta_2} \mathbf{wf}$*, then* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \preceq \delta_2$*.*
2. *If* $\Gamma = \Gamma_1, \blacktriangleleft^{\gamma_2}_{k}, \Gamma_2$ *and* $\Psi \mid^{\delta_1} \Gamma_1 \vdash^{\ldots,\delta_2} \mathbf{wf}$*, then* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \preceq \delta_2$*.*
3. *If* $\Gamma = \Gamma_1, [\gamma'_1 :\succeq \gamma''_1, \ldots, \gamma'_k :\succeq \gamma''_k]^{\gamma_2}, \Gamma_2$ *and* $\Psi \mid^{\delta_1} \Gamma_1 \vdash^{\ldots,\delta_2} \mathbf{wf}$*, then* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \preceq \delta_2$*.*
4. *If* $\Gamma = \Gamma_1, [\gamma'_1 :\succeq \gamma''_1, \ldots, \gamma'_k :\succeq \gamma''_k]^{\gamma'_0}, \Gamma_2$ *and* $\gamma'_i = \gamma_2$ *for some* i *with* $1 \leq i \leq k$*, then* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \preceq \gamma''_i$*.*

Proof. By induction on the given derivation. $\square$

Lemma 9 (Inversion for $\subseteq$ w.r.t. Reserved Names Typing)**.** *If* $\Psi \vdash \gamma_1 \subseteq \gamma_2$ *and* $\gamma_1 \neq \gamma_2$*, then the following hold:*

1. *If* $x : A \mathbin{@} \gamma_2 :\succeq \delta_1 \in \Psi$*, then* $\Psi \vdash \gamma_1 \subseteq \delta_1$*.*
2. *If* $\gamma_2(:\succeq \delta_1 / :\supseteq \delta_2) \in \Psi$*, then* $\Psi \vdash \gamma_1 \subseteq \delta_1$ *or* $\Psi \vdash \gamma_1 \subseteq \delta_2$*.*
3. *If* $[\delta_0 :\succeq \delta'_0, \ldots, \delta_k :\succeq \delta'_k] \in \Psi$ *and* $\gamma_2 = \delta_i$ *for some* i *with* $0 \leq i < k$*, then* $\Psi \vdash \gamma_1 \subseteq \delta'_i$ *or* $\Psi \vdash \gamma_1 \subseteq \delta_{i+1}$*.*
4. *If* $[\delta_0 :\succeq \delta'_0, \ldots, \delta_k :\succeq \delta'_k] \in \Psi$ *and* $\gamma_2 = \delta_k$*, then* $\Psi \vdash \gamma_1 \subseteq \delta'_k$*.*

Proof. By induction on the given derivation. $\square$

Lemma 10 (Inversion for $\subseteq$ w.r.t. Context)**.** *If* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \subseteq \gamma_2$*,* $\gamma_1 \neq \gamma_2$ *and* $\gamma_2 \in \mathbf{Dom}_C(\Gamma)$*, then the following hold:*

1. *If* $\Gamma = \Gamma_1, x{:}^{\gamma_2} A, \Gamma_2$ *and* $\Psi \mid^{\delta_1} \Gamma_1 \vdash^{\ldots,\delta_2} \mathbf{wf}$*, then* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \subseteq \delta_2$*.*
2. *If* $\Gamma = \Gamma_1, \blacktriangleleft^{\gamma_2}_{k_1}, \Gamma_2$ *and* $\Psi \mid^{\delta_1} \Gamma_1 \vdash^{\delta'_1,\ldots,\delta'_{k_2}} \mathbf{wf}$*, then* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \subseteq \delta'_{k_2}$ *or* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \subseteq \delta'_{k_2-k_1}$*.*
3. *If* $\Gamma = \Gamma_1, [\gamma'_1 :\succeq \gamma''_1, \ldots, \gamma'_k :\succeq \gamma''_k]^{\gamma_2}, \Gamma_2$ *and* $\Psi \mid^{\delta_1} \Gamma_1 \vdash^{\ldots,\delta_2} \mathbf{wf}$*, then* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \subseteq \delta_2$*.*
4. *If* $\Gamma = \Gamma_1, [\gamma'_1 :\succeq \gamma''_1, \ldots, \gamma'_k :\succeq \gamma''_k]^{\gamma'_0}, \Gamma_2$ *and* $\gamma'_i = \gamma_2$ *for some* i *with* $1 \leq i < k$*, then* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \subseteq \gamma''_i$ *or* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \subseteq \gamma'_{i+1}$*.*
5. *If* $\Gamma = \Gamma_1, [\gamma'_1 :\succeq \gamma''_1, \ldots, \gamma'_k :\succeq \gamma''_k]^{\gamma'_0}, \Gamma_2$ *and* $\gamma'_i = \gamma_2$ *for some* i *with* $1 \leq i \leq k$*, then* $\Psi \mid^{\delta_1} \Gamma \vdash \gamma_1 \subseteq \gamma''_i$*.*

Proof. By induction on the given derivation. $\square$

3.3 Structural Properties

Lemma 11 (Weakening with regard to Reserved Names Typing)**.** *Suppose* $\Psi_1 \subseteq \Psi_2$ *and* $\vdash \Psi_2$ $\mathbf{wf}$*. Then the following hold:*

1. *If* $\Psi_1 \vdash \overrightarrow{\gamma}^+ \,\mathbf{wf}$ *and* $\vdash \Psi_2$ $\mathbf{wf}$*, then* $\Psi_2 \vdash \overrightarrow{\gamma}^+ \,\mathbf{wf}$*.*
2. *If* $\Psi_1 \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_2}^+} A$ $\mathbf{wf}$ *and* $\Psi_2 \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_2}^+}$ $\mathbf{wf}$*, then* $\Psi_2 \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_2}^+} A$ $\mathbf{wf}$*.*
3. *If* $\Psi_1 \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash \gamma_2 \preceq \gamma_3$ *and* $\Psi_2 \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_2}^+}$ $\mathbf{wf}$*, then* $\Psi_2 \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash \gamma_2 \preceq \gamma_3$*.*
4. *If* $\Psi_1 \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash \gamma_2 \subseteq \gamma_3$ *and* $\Psi_2 \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_2}^+}$ $\mathbf{wf}$*, then* $\Psi_2 \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash \gamma_2 \subseteq \gamma_3$*.*
5. *If* $\Psi_1 \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} e : A$ *and* $\Psi_2 \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_2}^+}$ $\mathbf{wf}$*, then* $\Psi_2 \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} e : A$*.*
6. *If* $\Psi_1 \vdash \Theta$ $\mathbf{wf}$*, then* $\Psi_2 \vdash \Theta$ $\mathbf{wf}$*.*
7. *If* $\Psi_1 \vdash S \colon \Theta$*, then* $\Psi_2 \vdash S \colon \Theta$*.*
8. *If* $\Psi_1 \mid \Theta \vdash^{\gamma} v : A$ $\mathbf{val}$*, then* $\Psi_2 \mid \Theta \vdash^{\gamma} v : A$ $\mathbf{val}$*.*
9. *If* $\Psi_1 \mid \Theta \vdash^{\gamma} E$ $\mathbf{wf}$*, then* $\Psi_2 \mid \Theta \vdash^{\gamma} E$ $\mathbf{wf}$*.*

Proof. By mutual induction on the given derivation. □

Lemma 12 (Weakening with regard to Store Typing)**.** *Suppose* $\Psi \vdash \Theta_2$ $\mathbf{wf}$ *and* $\Theta_1 \subseteq \Theta_2$*. Then the following hold:*

1. *If* $\Psi \mid \Theta_1 \vdash^{\gamma} v : A$ $\mathbf{val}$*, then* $\Psi \mid \Theta_2 \vdash^{\gamma} v : A$ $\mathbf{val}$*.*
2. *If* $\Psi \mid \Theta_1 \vdash^{\gamma} E$ $\mathbf{wf}$*, then* $\Psi \mid \Theta_2 \vdash^{\gamma} E$ $\mathbf{wf}$*.*

Proof. By mutual induction on the given derivation. □

Lemma 13 (Monotonicity with regard to Reserved Names Typing)**.** *Suppose* $\Psi \vdash \overrightarrow{\delta_1}^+, \delta_2\, \mathbf{wf}$ *and* $\Psi \vdash \gamma_0 \preceq \delta_2$*. If* $\Psi \mid^{\gamma_0} \Gamma \vdash^{\overrightarrow{\gamma_1}^+}$ $\mathbf{wf}$*, then there exists* $\overrightarrow{\delta_3}^+$ *such that* $\Psi \mid^{\overrightarrow{\delta_1}^+, \delta_2} \Gamma \vdash^{\overrightarrow{\delta_3}^+}$ $\mathbf{wf}$*. For such* $\overrightarrow{\delta_3}^+$*, the following hold:*

1. *If* $\Psi \mid^{\gamma_0} \Gamma \vdash^{\overrightarrow{\gamma_1}^+} A$ $\mathbf{wf}$*, then* $\Psi \mid^{\overrightarrow{\delta_1}^+, \delta_2} \Gamma \vdash^{\overrightarrow{\delta_3}^+} A$ $\mathbf{wf}$*.*
2. *If* $\Psi \mid^{\gamma_0} \Gamma \vdash \gamma_1 \preceq \gamma_2$*, then* $\Psi \mid^{\overrightarrow{\delta_1}^+, \delta_2} \Gamma \vdash \gamma_1 \preceq \gamma_2$*.*
3. *If* $\Psi \mid^{\gamma_0} \Gamma \vdash \gamma_1 \subseteq \gamma_2$*, then* $\Psi \mid^{\overrightarrow{\delta_1}^+, \delta_2} \Gamma \vdash \gamma_1 \subseteq \gamma_2$*.*
4. *If* $\Psi \mid^{\gamma_0} \Gamma \vdash_R^{\overrightarrow{\gamma_1}^+} e : A$*, then* $\Psi \mid^{\overrightarrow{\delta_1}^+, \delta_2} \Gamma \vdash_R^{\overrightarrow{\delta_3}^+} e : A$*.*

Proof. By mutual induction on the given derivation. □

Lemma 14 (Weakening with regard to Context)**.**

1. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash \delta_1 \preceq \delta_2$ *and* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2 \vdash^{\cdots}$ $\mathbf{wf}$*, then* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2 \vdash \delta_1 \preceq \delta_2$*.*
2. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash \delta_1 \subseteq \delta_2$ *and* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2 \vdash^{\cdots}$ $\mathbf{wf}$*, then* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2 \vdash \delta_1 \subseteq \delta_2$*.*

Proof. By mutual induction on the given derivation. □

Lemma 15 (Strengthening with regard to Context)**.**

1. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2 \vdash \delta_1 \preceq \delta_2$ *and* $\delta_2 \notin \mathbf{Dom}_C(\Gamma_2)$*, then* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash \delta_1 \preceq \delta_2$*.*
2. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2 \vdash \delta_1 \subseteq \delta_2$ *and* $\delta_2 \notin \mathbf{Dom}_C(\Gamma_2)$*, then* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash \delta_1 \subseteq \delta_2$*.*

Proof. By mutual induction on the given derivation. □

Lemma 16 (Classifier Substitution Lemma for Polymorphic Classifier Declarations)**.** *Let* $\Gamma_0 = [\gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k]^{\gamma_0}$ *and* $\sigma = \gamma_0 \coloneqq \delta'_0, \ldots, \gamma_k \coloneqq \delta'_k$*. Suppose* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1 \vdash^{\overrightarrow{\delta_3},\delta'_0} \mathbf{wf}$*,* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1 \vdash \gamma'_i \preceq \delta'_i$ *for* $1 \leq i \leq k$*, and* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1 \vdash \delta'_{i+1} \subseteq \delta'_i$ *for* $0 \leq i < k$*. Then the following hold:*

1. *If* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1, \Gamma_0, \Gamma_2 \vdash^{\overrightarrow{\delta_2}+} \mathbf{wf}$*, then* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1, \Gamma_2[\sigma] \vdash^{\overrightarrow{\delta_2}+[\sigma]} \mathbf{wf}$*.*
2. *If* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1, \Gamma_0, \Gamma_2 \vdash^{\overrightarrow{\delta_2}+} A\ \mathbf{wf}$*, then* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1, \Gamma_2[\sigma] \vdash^{\overrightarrow{\delta_2}+[\sigma]} A[\sigma]\ \mathbf{wf}$*.*
3. *If* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1, \Gamma_0, \Gamma_2 \vdash \delta''_1 \preceq \delta''_2$*, then* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1, \Gamma_2[\sigma] \vdash \delta''_1[\sigma] \preceq \delta''_2[\sigma]$*.*
4. *If* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1, \Gamma_0, \Gamma_2 \vdash \delta''_1 \subseteq \delta''_2$*, then* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1, \Gamma_2[\sigma] \vdash \delta''_1[\sigma] \subseteq \delta''_2[\sigma]$*.*
5. *If* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1, \Gamma_0, \Gamma_2 \vdash_R^{\overrightarrow{\delta_2}+} e : A$*, then* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1, \Gamma_2[\sigma] \vdash_R^{\overrightarrow{\delta_2}+[\sigma]} e : A[\sigma]$*.*

Proof. By mutual induction on the given derivation. □

Lemma 17. *If* $\Psi \mid^{\gamma_1} \blacktriangleright^{\gamma_2} \vdash_{\mathbf{id}}^{\gamma_1,\gamma_2} e^0 : A$*, then* $\Psi \mid^{\gamma_2} \varepsilon \vdash_{\mathbf{id}}^{\gamma_2} e^0 : A$*.*

Proof. By induction on the given derivation. □

3.4 Normal Form of Expression Typing Derivations

Definition 11. *We define an additional rule R-T-Appcls-NonIntro for expression typing.*

R-T-Appcls-NonIntro

$$\frac{\begin{array}{c}\Psi \mid^{\overrightarrow{\delta'_1}+} \Gamma \vdash_R^{\overrightarrow{\delta'_2},\delta_0} e : \forall \gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k.A \\ \Psi \mid^{\overrightarrow{\delta'_1}+} \Gamma \vdash \gamma'_i \preceq \delta_i \text{ for } 1 \leq i \leq k \qquad \Psi \mid^{\overrightarrow{\delta'_1}+} \Gamma \vdash \delta_{i+1} \subseteq \delta_i \text{ for } 0 \leq i < k \qquad e \text{ is not an introduction form}\end{array}}{\Psi \mid^{\overrightarrow{\delta'_1}+} \Gamma \vdash_R^{\overrightarrow{\delta'_2},\delta_0} e : A[\gamma_1 \coloneqq \delta_1, \ldots, \gamma_k \coloneqq \delta_k]}$$

If an expression typing derivation contains no use of R-T-Appcls, then we say it is in normal form*.*

Lemma 18 (Classifier Substitution Lemma for Polymorphic Classifier Declarations)**.** *Let* $\Gamma_0 = [\gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k]^{\gamma_0}$ *and* $\sigma = \gamma_0 \coloneqq \delta'_0, \ldots, \gamma_k \coloneqq \delta'_k$*. Suppose* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1 \vdash^{\overrightarrow{\delta_3},\delta'_0} \mathbf{wf}$*,* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1 \vdash \gamma'_i \preceq \delta'_i$ *for* $1 \leq i \leq k$*, and* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1 \vdash \delta'_{i+1} \subseteq \delta'_i$ *for* $0 \leq i < k$*. If there is a normal derivation for* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1, \Gamma_0, \Gamma_2 \vdash_R^{\overrightarrow{\delta_2}+} e : A$*, then there is also a normal derivation for* $\Psi \mid^{\overrightarrow{\delta_1}+} \Gamma_1, \Gamma_2[\sigma] \vdash_R^{\overrightarrow{\delta_2}+[\sigma]} e : A[\sigma]$*.*

Proof. By induction on the given derivation, as done in Lemma 16. □

Lemma 19 (Normal Form of Expression Typing Derivations)**.** *If an expression typing judgment is derivable, then it is derivable with a normal form derivation. Namely, R-T-Appcls can be safely restricted to R-T-Appcls-NonIntro.*

Proof. By induction on the given derivation. The only non-trivial case is when the last rule is R-T-Appcls and the expression is an introduction form. By the induction hypothesis, the derivation of the first premise of R-T-Appcls can be taken to be in normal form. Since the expression is an introduction form, this normal-form derivation must end with R-T-Polycls. We can therefore apply Lemma 18 to obtain a normal-form derivation of the same typing judgment. □

Convention 1. *In the rest of this note, we assume that all expression typing derivations are in normal form. Accordingly, we use R-T-Appcls to refer to R-T-Appcls-NonIntro, unless stated otherwise.*

3.5 Upper Bound of Classifiers

Lemma 20. $\mathbf{Ctx}^{k_1,k_2} \subseteq \mathbf{Ctx}^{k_1+k_3,k_2+k_3}$ *for any* $k_3 > 0$.

Proof. By induction on the structure of contexts. □

Lemma 21. $\Gamma_1^{k_1,k_2}, \Gamma_2^{k_2,k_3} \in \mathbf{Ctx}^{k_1,k_3}$.

Proof. By induction on the structure of $\Gamma_1^{k_1,k_2}$. □

Lemma 22 (Origins of Output Classifiers)**.** *If*

- $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash^{\overrightarrow{\gamma_2},\delta_0,\ldots,\delta_{k_1}} \mathbf{wf}$,
- $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash^{\overrightarrow{\gamma_2},\overrightarrow{\gamma_3}^+} \mathbf{wf}$, *and*
- $\gamma_4 \in \{\overrightarrow{\gamma_3}^+\}$,

then $\gamma_4 \in \{\delta_0, \ldots, \delta_{k_1}\} \cup \mathbf{Dom}_C(\Gamma_2^{k_1,k_2}) \cup \mathbf{FC}(\Gamma_2^{k_1,k_2})$.

Proof. Straightforward. □

Lemma 23. *If*

- $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash^{\overrightarrow{\gamma_2},\delta_0,\ldots,\delta_{k_1}} \mathbf{wf}$,
- $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_3 \subseteq \gamma_4$,
- $\gamma_3 \in \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma_1)$, *and*
- $\gamma_4 \in \{\delta_0, \ldots, \delta_{k_1}\} \cup \mathbf{Dom}_C(\Gamma_2^{k_1,k_2}) \cup \mathbf{FC}(\Gamma_2^{k_1,k_2})$,

then there exists γ_5 *such that*

- $\gamma_5 \in \{\delta_0, \ldots, \delta_{k_1}\} \cup \mathbf{FC}(\Gamma_2^{k_1,k_2})$,
- $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_5 \subseteq \gamma_4$ *and*
- $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_3 \subseteq \gamma_5$

Proof. By induction with respect to $\mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{k_1,k_2}), \gamma_4)$.

If $\gamma_4 \in \{\delta_0, \ldots, \delta_{k_1}\} \cup \mathbf{FC}(\Gamma_2^{k_1,k_2})$, then take $\gamma_5 = \gamma_4$. The two required visibility judgments follow by reflexivity and the given derivation.

We consider the remaining case where $\gamma_4 \in \mathbf{Dom}_C(\Gamma_2^{k_1,k_2})$. If $\gamma_3 = \gamma_4$, then this contradicts the freshness/disjointness guaranteed by context well-formedness, because $\gamma_3 \in \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma_1)$. Hence, we assume $\gamma_3 \neq \gamma_4$ and perform case analysis on how γ_4 is introduced in $\Gamma_2^{k_1,k_2}$.

Case $\Gamma_2^{k_1,k_2} = \Gamma_3^{k_1,k_3}, x{:}^{\gamma_4} A, \Gamma_4$**:** By regularity, we have $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_3} \vdash^{\ldots,\delta''} \mathbf{wf}$ for some δ''. By Lemma 10, $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_3 \subseteq \delta''$. Moreover, by Lemma 22, $\delta'' \in \{\delta_0, \ldots, \delta_{k_1}\} \cup \mathbf{Dom}_C(\Gamma_2^{k_1,k_2}) \cup \mathbf{FC}(\Gamma_2^{k_1,k_2})$. Since $\mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{k_1,k_2}), \delta'') < \mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{k_1,k_2}), \gamma_4)$, the induction hypothesis gives some γ_5 such that

$$\gamma_5 \in \{\delta_0, \ldots, \delta_{k_1}\} \cup \mathbf{FC}(\Gamma_2^{k_1,k_2}), \qquad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_5 \subseteq \delta'', \quad \text{and} \quad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_3 \subseteq \gamma_5.$$

Reapplying the variable context rule gives $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \delta'' \subseteq \gamma_4$. Thus $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_5 \subseteq \gamma_4$ by R-$\subseteq$-Trans.

Case $\Gamma_2^{k_1,k_2} = \Gamma_3^{k_1,k_3}, \blacktriangleleft_{k_4}^{\gamma_4}, \Gamma_4$**:** By regularity, we have $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_3} \vdash^{\dots,\delta'_0,\dots,\delta'_{k_4}}$ **wf**. By Lemma 10, either $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_3 \subseteq \delta'_0$ or $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_3 \subseteq \delta'_{k_4}$. Let δ'' be the right-hand classifier obtained in this way. By Lemma 22, $\delta'' \in \{\delta_0, \dots, \delta_{k_1}\} \cup \mathbf{Dom}_C(\Gamma_2^{k_1,k_2}) \cup \mathbf{FC}(\Gamma_2^{k_1,k_2})$. Since $\mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{k_1,k_2}), \delta'') < \mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{k_1,k_2}), \gamma_4)$, the induction hypothesis gives some γ_5 such that

$$\gamma_5 \in \{\delta_0, \dots, \delta_{k_1}\} \cup \mathbf{FC}(\Gamma_2^{k_1,k_2}), \qquad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_5 \subseteq \delta'', \quad \text{and} \quad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_3 \subseteq \gamma_5.$$

Reapplying the corresponding $\blacktriangleleft$ context rule gives $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \delta'' \subseteq \gamma_4$. Thus $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_5 \subseteq \gamma_4$ by R-$\subseteq$-Trans.

Case $\Gamma_2^{k_1,k_2} = \Gamma_3^{k_1,k_3}, [\gamma'_1 :\succeq \gamma''_1, \dots, \gamma'_k :\succeq \gamma''_k]^{\gamma_4}, \Gamma_4$**:** By regularity, we have $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_3} \vdash^{\dots,\delta''}$ **wf** for some δ''. By Lemma 10, the derivation factors through the output classifier δ''. By Lemma 22, $\delta'' \in \{\delta_0, \dots, \delta_{k_1}\} \cup \mathbf{Dom}_C(\Gamma_2^{k_1,k_2}) \cup \mathbf{FC}(\Gamma_2^{k_1,k_2})$. Since $\mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{k_1,k_2}), \delta'') < \mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{k_1,k_2}), \gamma_4)$, the induction hypothesis gives some γ_5 such that

$$\gamma_5 \in \{\delta_0, \dots, \delta_{k_1}\} \cup \mathbf{FC}(\Gamma_2^{k_1,k_2}), \qquad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_5 \subseteq \delta'', \quad \text{and} \quad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_3 \subseteq \gamma_5.$$

Reapplying the corresponding polymorphic-classifier context rule gives $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \delta'' \subseteq \gamma_4$. Thus $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_5 \subseteq \gamma_4$ by R-$\subseteq$-Trans.

Case $\Gamma_2^{k_1,k_2} = \Gamma_3^{k_1,k_3}, [\gamma'_1 :\succeq \gamma''_1, \dots, \gamma'_k :\succeq \gamma''_k]^{\gamma'_0}, \Gamma_4$ and $\gamma_4 = \gamma'_i$ for some i with $1 \le i \le k$**:** By Lemma 10, the derivation factors through either γ''_i or, when $i < k$, γ'_{i+1}. Let δ'' be the right-hand classifier obtained in this way. The membership condition $\delta'' \in \{\delta_0, \dots, \delta_{k_1}\} \cup \mathbf{Dom}_C(\Gamma_2^{k_1,k_2}) \cup \mathbf{FC}(\Gamma_2^{k_1,k_2})$ follows from the definitions of $\mathbf{Dom}_C(\Gamma_2^{k_1,k_2})$ and $\mathbf{FC}(\Gamma_2^{k_1,k_2})$. Since $\mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{k_1,k_2}), \delta'') < \mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{k_1,k_2}), \gamma_4)$, the induction hypothesis gives some γ_5 such that

$$\gamma_5 \in \{\delta_0, \dots, \delta_{k_1}\} \cup \mathbf{FC}(\Gamma_2^{k_1,k_2}), \qquad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_5 \subseteq \delta'', \quad \text{and} \quad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_3 \subseteq \gamma_5.$$

Reapplying the corresponding polymorphic-classifier context rule gives $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \delta'' \subseteq \gamma_4$. Thus $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_5 \subseteq \gamma_4$ by R-$\subseteq$-Trans.

□

Lemma 24 (Upper Bound of Classifiers)**.** *Suppose*

- $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash^{\overrightarrow{\gamma_2},\delta_0,\dots,\delta_{k_1}}$ **wf** *and*
- $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash^{\overrightarrow{\gamma_4}^+}$ **wf**.

If $\gamma_5 \in \mathbf{FC}(\Gamma_2^{k_1,k_2})$*, then* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash \gamma_5 \subseteq \delta_i$ *for some* i *such that* $0 \le i \le k_1$*.*

Proof. By induction with respect to the size of $\Gamma_2^{k_1,k_2}$. By the definition of $\mathbf{FC}(\Gamma_2^{k_1,k_2})$, it suffices to consider the cases in which the occurrence of γ_5 is introduced by the last context constructor of $\Gamma_2^{k_1,k_2}$. If the occurrence comes from a proper subcontext, the result follows from the induction hypothesis and Lemma 14.

Case $\Gamma_2^{k_1,k_2} = \Gamma_3^{k_1,k_2-1}, \blacktriangleright^{\gamma_5}$**:** By regularity, we have $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_2-1} \vdash^{\dots,\gamma_6}$ **wf** for some γ_6. The well-formedness of $\Gamma_1, \Gamma_2^{k_1,k_2}$ gives $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_2-1} \vdash \gamma_5 \subseteq \gamma_6$. By Lemma 22, $\gamma_6 \in \{\delta_0, \dots, \delta_{k_1}\} \cup \mathbf{Dom}_C(\Gamma_3^{k_1,k_2-1}) \cup \mathbf{FC}(\Gamma_3^{k_1,k_2-1})$. Also $\gamma_5 \in \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma_1)$ from $\gamma_5 \in \mathbf{FC}(\Gamma_2^{k_1,k_2})$. Hence Lemma 23 gives some γ_7 such that

$$\gamma_7 \in \{\delta_0, \dots, \delta_{k_1}\} \cup \mathbf{FC}(\Gamma_3^{k_1,k_2-1}), \qquad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_2-1} \vdash \gamma_7 \subseteq \gamma_6, \quad \text{and} \quad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_2-1} \vdash \gamma_5 \subseteq \gamma_7.$$

If $\gamma_7 \in \{\delta_0, \dots, \delta_{k_1}\}$, choose i such that $\delta_i = \gamma_7$. By Lemma 14, the judgment $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_5 \subseteq \delta_i$ gives the desired judgment for δ_i.

Otherwise, $\gamma_7 \in \mathbf{FC}(\Gamma_3^{k_1,k_2-1})$. By the induction hypothesis, there exists some j with $0 \le j \le k_1$ such that $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_2-1} \vdash \gamma_7 \subseteq \delta_j$. By transitivity, we obtain $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_2-1} \vdash \gamma_5 \subseteq \delta_j$. Then, by Lemma 14, $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_5 \subseteq \delta_j$. Since $\delta_j \in \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma_1)$, applying Lemma 15 yields $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash \gamma_5 \subseteq \delta_j$. Hence, we may take $i = j$.

Case $\Gamma_2^{k_1,k_2} = \Gamma_3^{k_1,k_2}, [\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_{k_3} :\succeq \gamma'_{k_3}]^{\delta'_0}$ and $\gamma_5 = \gamma'_i$ with $1 \le i \le k_3$**:** By regularity, we have $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_2} \vdash^{\ldots,\gamma'_0}$ **wf**. The well-formedness of $\Gamma_1, \Gamma_2^{k_1,k_2}$ gives $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_2} \vdash \gamma'_i \subseteq \gamma'_0$. By Lemma 22, $\gamma'_0 \in \{\delta_0, \ldots, \delta_{k_1}\} \cup \mathbf{Dom}_C(\Gamma_3^{k_1,k_2}) \cup \mathbf{FC}(\Gamma_3^{k_1,k_2})$. Also $\gamma_5 \in \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma_1)$ from $\gamma_5 \in \mathbf{FC}(\Gamma_2^{k_1,k_2})$. Hence Lemma 23 gives some γ_6 such that

$$\gamma_6 \in \{\delta_0, \ldots, \delta_{k_1}\} \cup \mathbf{FC}(\Gamma_3^{k_1,k_2}), \qquad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_2} \vdash \gamma_6 \subseteq \gamma'_0, \quad \text{and} \quad \Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_2} \vdash \gamma_5 \subseteq \gamma_6.$$

If $\gamma_6 \in \{\delta_0, \ldots, \delta_{k_1}\}$, choose i such that $\delta_i = \gamma_6$. By Lemma 14, the judgment $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_5 \subseteq \delta_i$ gives the desired judgment for δ_i.

Otherwise $\gamma_6 \in \mathbf{FC}(\Gamma_3^{k_1,k_2})$, and the induction hypothesis gives some $0 \le j \le k_1$ with $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_2} \vdash \gamma_6 \subseteq \delta_j$. By transitivity, $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_3^{k_1,k_2} \vdash \gamma_5 \subseteq \delta_j$. Then $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1, \Gamma_2^{k_1,k_2} \vdash \gamma_5 \subseteq \delta_j$ by Lemma 14. Thus we can take $i = j$.

In either case, since $\delta_i \in \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma_1)$, applying Lemma 15 yields $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma_1 \vdash \gamma_5 \subseteq \delta_i$.

□

Corollary 1 (Upper Bound of Classifiers Case 1)**.** *Suppose* $\Psi \mid^{\gamma_1} \Gamma_1 \vdash^{\overrightarrow{\gamma_2},\gamma_3}$ **wf** *and* $\Psi \mid^{\gamma_1} \Gamma_1, \Gamma_2^{0,k} \vdash^{\overrightarrow{\gamma_4}^+}$ **wf**. *If* $\gamma_5 \in \mathbf{FC}(\Gamma_2^{0,k})$, *then* $\Psi \mid^{\gamma_1} \Gamma_1 \vdash \gamma_5 \subseteq \gamma_3$.

Corollary 2 (Upper Bound of Classifiers Case 2)**.** *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash^{\overrightarrow{\gamma_2}^+}$ **wf**, *and* $\gamma_3 \in \mathbf{FC}(\Gamma)$, *then* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash \gamma_3 \subseteq \gamma_4$ *for some* $\gamma_4 \in \{\overrightarrow{\gamma_1}^+\}$.

3.6 Properties of Value Typing

Lemma 25. *Let* $\Gamma_1 = [\varrho_1]^{\gamma_1}, \ldots, [\varrho_{k_1}]^{\gamma_{k_1}}$ *and* $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_1 \subseteq \gamma_{k_1}$ *where* $k_1 \ge 0$. *Suppose* $\gamma_i \notin \mathbf{FC}(\varrho_{i+1}) \cup \cdots \cup \mathbf{FC}(\varrho_{k_1}) \cup \{\delta_1\}$ *for* $1 \le i \le k_1$. *If* $\delta_1 \in \mathbf{Dom}_C(\Gamma_1)$ *and* $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_2 \subseteq \delta_1$ *then* $\delta_2 \notin \{\gamma_1, \ldots, \gamma_{k_1}\}$.

Proof. By induction on $\mathbf{deg}(\Psi, \Gamma_1, \delta_1)$. The freshness assumption gives $\delta_1 \notin \{\gamma_1, \ldots, \gamma_{k_1}\}$. Since $\delta_1 \in \mathbf{Dom}_C(\Gamma_1)$, there exist k_2, k_3, and k_4 such that $\varrho_{k_2} = \gamma'_1 :\succeq \gamma''_1, \ldots, \gamma'_{k_3} :\succeq \gamma''_{k_3}$, $\delta_1 = \gamma'_{k_4}$, and $1 \le k_4 \le k_3$.

First suppose $k_4 < k_3$. By Lemma 10, either $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_2 \subseteq \gamma''_{k_4}$ or $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_2 \subseteq \gamma'_{k_4+1}$. Let δ_3 be the right-hand classifier obtained in this way. Then $\delta_3 \in \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma_1)$ by Lemma 1.

If $\delta_3 \in \mathbf{Dom}_C(\Psi)$, then $\delta_2 \in \mathbf{Dom}_C(\Psi)$ follows from $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_2 \subseteq \delta_3$ and Lemma 3. Thus $\delta_2 \notin \{\gamma_1, \ldots, \gamma_{k_1}\}$ by the disjointness of $\mathbf{Dom}_C(\Psi)$ and $\mathbf{Dom}_C(\Gamma_1)$.

If $\delta_3 \in \mathbf{Dom}_C(\Gamma_1)$, then ϱ_{k_2} gives $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_3 \subseteq \delta_1$. Together with $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_1 \subseteq \gamma_{k_1}$, transitivity yields $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_3 \subseteq \gamma_{k_1}$.

Moreover, the original non-occurrence assumption implies

$$\gamma_i \notin \mathbf{FC}(\varrho_{i+1}) \cup \cdots \cup \mathbf{FC}(\varrho_{k_1}) \cup \{\delta_3\}.$$

Here, we use $\delta_3 \in \mathbf{FC}(\varrho_{k_2})$ when $\delta_3 = \gamma''_{k_4}$, and $\gamma_i \ne \gamma'_{k_4+1}$ when $\delta_3 = \gamma'_{k_4+1}$.

Finally, by Lemma 3,

$$\mathbf{deg}(\Psi, \Gamma_1, \delta_3) < \mathbf{deg}(\Psi, \Gamma_1, \delta_1).$$

Therefore, applying the induction hypothesis to δ_3 yields $\delta_2 \notin \{\gamma_1, \ldots, \gamma_{k_1}\}$.

The case $k_4 = k_3$ is analogous, using the conclusion $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_2 \subseteq \gamma''_{k_4}$ obtained from Lemma 10. □

Lemma 26. *Let* $\Gamma_1 = [\varrho_1]^{\gamma_1}, \ldots, [\varrho_{k_1}]^{\gamma_{k_1}}, \blacktriangleright^{\delta_1}$ *where* $k_1 \ge 0$ *and* $\gamma_i \notin \mathbf{FC}(\varrho_{i+1}) \cup \cdots \cup \mathbf{FC}(\varrho_{k_1}) \cup \{\delta_1\}$ *for* $1 \le i \le k_1$. *If* $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash^{\gamma_{k_1}[\gamma_0 := \gamma'], \overrightarrow{\delta_2}^+}$ **wf**, *then the following hold:*

1. *If* $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta'_1 \preceq \delta'_2$ *and either* $\delta'_2 \in \mathbf{Dom}_C(\Gamma_2^{0,k_2})$ *or* $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta'_2 \subseteq \delta_1$, *then* $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta'_1 \preceq \delta'_2$.
2. *If* $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta'_1 \subseteq \delta'_2$ *and either* $\delta'_2 \in \mathbf{Dom}_C(\Gamma_2^{0,k_2})$ *or* $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta'_2 \subseteq \delta_1$, *then* $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta'_1 \subseteq \delta'_2$.
3. *If* $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash^{\gamma_{k_1}, \overrightarrow{\delta_2}^+} A$ **wf**, *then* $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash^{\gamma_{k_1}[\gamma_0 := \gamma'], \overrightarrow{\delta_2}^+} A$ **wf**
4. *If* $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash_{\mathbf{id}}^{\gamma_{k_1}, \overrightarrow{\delta_2}^+} e^{k_2} : A$, *then* $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash_{\mathbf{id}}^{\gamma_{k_1}[\gamma_0 := \gamma'], \overrightarrow{\delta_2}^+} e^{k_2} : A$.

Proof. Let $\Gamma = \Gamma_1, \Gamma_2^{0,k_2}$. We prove the four statements by mutual induction on the derivations appearing in their premises.

Item 1. The proof is analogous to that of Item 2, using the rules and lemmas for $\preceq$ in place of those for $\subseteq$.

Item 2. By induction on $\mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{0,k_2}), \delta_2')$. Since $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_1' \subseteq \delta_2'$ is derivable, the classifier δ_2' is declared in Ψ, Γ_1, or Γ_2^{0,k_2} by Lemma 1.

Case $\delta_2' \in \mathbf{Dom}_C(\Gamma_2^{0,k_2})$**:** We proceed by case analysis using Lemma 10. We spell out the case where δ_2' is introduced by a $\blacktriangleleft$ entry in Γ_2^{0,k_2}.

Suppose $\Gamma_2^{0,k_2} = \Gamma_3, \blacktriangleleft_{k_3}^{\delta_2'}, \Gamma_4$ and $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_3 \vdash^{\gamma_{k_1}, \delta_1'', \ldots, \delta_{k_4}''} \mathbf{wf}$. By Lemma 10, either $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_1' \subseteq \delta_{k_4}''$ or $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_1' \subseteq \delta_{k_4-k_3}''$.

In the former case, by Corollary 1, either $\delta_{k_4}'' \in \mathbf{Dom}_C(\Gamma_3)$ or $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_{k_4}'' \subseteq \delta_1$. As $\mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{0,k_2}), \delta_{k_4}'') < \mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{0,k_2}), \delta_2')$, we apply the induction hypothesis to δ_{k_4}'' and obtain $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_1' \subseteq \delta_{k_4}''$.

In the latter case, by Lemma 1, $\Gamma_3 = \Gamma_5, \Gamma_6$ for some Γ_5 and Γ_6 such that $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_5 \vdash^{\gamma_{k_1}, \delta_1'', \ldots, \delta_{k_4-k_3}''} \mathbf{wf}$. By Corollary 1, either $\delta_{k_4-k_3}'' \in \mathbf{Dom}_C(\Gamma_5)$ or $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_{k_4-k_3}'' \subseteq \delta_1$. $\delta_{k_4-k_3}'' \in \mathbf{Dom}_C(\Gamma_5)$ implies $\delta_{k_4-k_3}'' \in \mathbf{Dom}_C(\Gamma_3)$. As $\mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{0,k_2}), \delta_{k_4-k_3}'') < \mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{0,k_2}), \delta_2')$, we apply the induction hypothesis to $\delta_{k_4-k_3}''$ and obtain $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_1' \subseteq \delta_{k_4-k_3}''$.

Let δ_3 be δ_{k_4}'' or $\delta_{k_4-k_3}''$ for each case. Now we have $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_1' \subseteq \delta_3$. By inverting the well-formedness assumption for $\Gamma_1, \Gamma_2^{0,k_2}$ and its subcontext with Lemma 1, we have $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_3 \vdash^{\gamma_{k_1}[\gamma_0 := \gamma'], \delta_1'', \ldots, \delta_{k_4}''} \mathbf{wf}$.

If $\delta_3 = \delta_{k_4}''$, reapplying R-$\subseteq$-$\blacktriangleleft$-Ctx yields $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_3 \subseteq \delta_2'$. If $\delta_3 = \delta_{k_4-k_3}''$, the same judgment follows by R-$\preceq$-$\blacktriangleleft$-Ctx and R-$\subseteq$-Lift. Therefore, by R-$\subseteq$-Trans, $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_1' \subseteq \delta_2'$.

Case $\delta_2' \in \mathbf{Dom}_C(\Gamma_1)$**:** Since $\delta_2' \in \mathbf{Dom}_C(\Gamma_1)$, the alternative $\delta_2' \in \mathbf{Dom}_C(\Gamma_2^{0,k_2})$ is impossible by the disjointness of $\mathbf{Dom}_C(\Gamma_1)$ and $\mathbf{Dom}_C(\Gamma_2^{0,k_2})$. Hence the assumption gives $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_2' \subseteq \delta_1$.

Let $\Gamma_0 = [\varrho_1]^{\gamma_1}, \ldots, [\varrho_k]^{\gamma_{k_1}}$, so that $\Gamma_1 = \Gamma_0, \blacktriangleright^{\delta_1}$. We verify the premises of Lemma 25 for this Γ_0. First, by construction of Γ_1, $\Psi \mid^{\gamma_0} \Gamma_0 \vdash \delta_1 \subseteq \gamma_{k_1}$. Second, δ_1 must be declared in Ψ or in Γ_0, because $\blacktriangleright^{\delta_1}$ does not declare δ_1. The former is impossible: if $\delta_1 \in \mathbf{Dom}_C(\Psi)$, then Lemma 3 applied to $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_2' \subseteq \delta_1$ contradicts $\delta_2' \in \mathbf{Dom}_C(\Gamma_1)$. Thus $\delta_1 \in \mathbf{Dom}_C(\Gamma_0)$. Finally, the trailing $\blacktriangleright^{\delta_1}$ does not contribute to the derivation of $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_2' \subseteq \delta_1$, so $\Psi \mid^{\gamma_0} \Gamma_0 \vdash \delta_2' \subseteq \delta_1$. Therefore, by Lemma 25, we have $\delta_2' \notin \{\gamma_1, \ldots, \gamma_{k_1}\}$.

By Lemma 10 and $\delta_2' \notin \{\gamma_1, \ldots, \gamma_{k_1}\}$, we have $\varrho_{k_3} = \gamma_1'' :\succeq \delta_1'', \ldots, \gamma_{k_4}'' :\succeq \delta_{k_4}''$ and $\delta_2' = \gamma_{k_5}''$ for some k_5 such that $1 \leq k_5 \leq k_4$.

We consider the case $k_5 < k_4$; the case $k_5 = k_4$ is analogous. By Lemma 10, either $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_1' \subseteq \delta_{k_5}''$ or $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_1' \subseteq \gamma_{k_5+1}''$ holds. Let δ_3 be the right-hand classifier obtained in this way. The declaration ϱ_{k_3} gives $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_3 \subseteq \delta_2'$. By R-$\subseteq$-Trans, combining this with $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_2' \subseteq \delta_1$ yields $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_3 \subseteq \delta_1$.

Moreover, $\mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{0,k_2}), \delta_3) < \mathbf{deg}(\Psi, (\Gamma_1, \Gamma_2^{0,k_2}), \delta_2')$ by Lemma 3. Hence the induction hypothesis applies to δ_3, yielding $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_1' \subseteq \delta_3$.

By transitivity, combining $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_1' \subseteq \delta_3$ with $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_3 \subseteq \delta_2'$ we obtain $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_1' \subseteq \delta_2'$ as required.

Case $\delta_2' \in \mathbf{Dom}_C(\Psi)$**:** By Lemma 5, we have $\Psi \vdash \delta_1' \subseteq \delta_2'$. Moreover, the well-formedness assumption $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash^{\gamma_{k_1}[\gamma_0 := \gamma'], \overrightarrow{\delta_2}^{+}} A\ \mathbf{wf}$ gives $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash^{\cdots} \mathbf{wf}$. Hence, by Lemma 6, we obtain $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_1' \subseteq \delta_2'$.

Item 3. By induction on $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash^{\gamma_{k_1}, \overrightarrow{\delta_2}^{+}} A\ \mathbf{wf}$.

Item 4. By induction on $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash_{\mathbf{id}}^{\gamma_{k_1}, \overrightarrow{\delta_2}^{+}} e^{k_2} : A$. We cover major cases.

Case R-T-Var**:** By inversion of R-T-Var, we have $e^{k_2} = x$, $\overrightarrow{\delta_2}^+ = \overrightarrow{\delta_3}, \delta_4$, $\mathbf{lookup}\,((\Gamma_1, \Gamma_2^{0,k_2}), x) = \mathbf{Done}\,(A, \delta_5)$,

$$\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash^{\gamma_{k_1}, \overrightarrow{\delta_3}, \delta_4} \mathbf{wf}$$

and $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_5 \preceq \delta_4$. By Corollary 1, either $\delta_4 \in \mathbf{Dom}_C(\Gamma_2^{0,k_2})$ or $\Psi \mid^{\gamma_0} \Gamma_1 \vdash \delta_4 \subseteq \delta_1$. Hence, by Item 1, $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash \delta_5 \preceq \delta_4$. The required context well-formedness follows from the assumption $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash^{\gamma_{k_1}[\gamma_0 := \gamma'], \overrightarrow{\delta_3}, \delta_4} A\ \mathbf{wf}$. Therefore, by R-T-Var, we obtain $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash_{\mathbf{id}}^{\gamma_{k_1}[\gamma_0 := \gamma'], \overrightarrow{\delta_3}, \delta_4} x : A$.

Case R-T-Func**:** By inversion of R-T-Func, we have $e^{k_2} = \lambda x.\ {e_1}^{k_2}$, $A = A_1 \to A_2$, and $\overrightarrow{\delta_2}^+ = \overrightarrow{\delta_3}, \delta_4$. The derivation gives

$$\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2}, x{:}^{\delta_5} A_1 \vdash_{\mathbf{id}}^{\gamma_{k_1}, \overrightarrow{\delta_3}, \delta_5} {e_1}^{k_2} : A_2$$

together with $\delta_5 \notin \mathbf{FC}(A_2)$ and $\Psi \mid^{\gamma_0} \Gamma_1, \Gamma_2^{0,k_2} \vdash^{\gamma_{k_1}, \overrightarrow{\delta_3}, \delta_4} \mathbf{wf}$. By the induction hypothesis applied to the body derivation, we have

$$\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2}, x{:}^{\delta_5} A_1 \vdash_{\mathbf{id}}^{\gamma_{k_1}[\gamma_0 := \gamma'], \overrightarrow{\delta_3}, \delta_5} {e_1}^{k_2} : A_2.$$

From the assumption $\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash^{\gamma_{k_1}[\gamma_0 := \gamma'], \overrightarrow{\delta_3}, \delta_4} A_1 \to A_2\ \mathbf{wf}$, inversion gives the well-formedness premises needed for reapplying R-T-Func. The freshness premise required by R-T-Func is the same as in the original derivation, namely $\delta_5 \notin \mathbf{FC}(A_2)$. Therefore, by R-T-Func, we obtain

$$\Psi \mid^{\gamma'} \Gamma_1, \Gamma_2^{0,k_2} \vdash_{\mathbf{id}}^{\gamma_{k_1}[\gamma_0 := \gamma'], \overrightarrow{\delta_3}, \delta_4} \lambda x.\ {e_1}^{k_2} : A_1 \to A_2.$$

□

Lemma 27 (Value Portability)**.** *If* $\Psi \mid \Theta \vdash^{\gamma_1} v : A\ \mathbf{val}$*, then for any* γ_2 *such that* $\Psi \vdash^{\gamma_2} A\ \mathbf{wf}$*, we have* $\Psi \mid \Theta \vdash^{\gamma_2} v : A\ \mathbf{val}$.

Proof. We proceed by case analysis on the derivation of $\Psi \mid \Theta \vdash^{\gamma_1} v : A\ \mathbf{val}$.

Case VT-Int**:** Trivial.

Case VT-Clos-Func**:** By inversion of VT-Clos-Func, we have $v = \mathbf{clos}\,(E, R, \lambda x.\ e^0)$ and $A = \forall \varrho_1 \ldots \forall \varrho_k. A_1 \to A_2$. Moreover, the derivation gives

$$\Psi \mid^{\delta_0} \varepsilon \vdash_R^{\delta_0} \lambda x.\ e^0 : \forall \varrho_1 \ldots \forall \varrho_k. A_1 \to A_2$$

together with $\Psi \mid \Theta \vdash^{\delta_0} E\ \mathbf{wf}$. The assumption of the lemma gives $\Psi \vdash^{\gamma_2} \forall \varrho_1 \ldots \forall \varrho_k. A_1 \to A_2\ \mathbf{wf}$. Therefore, by reapplying VT-Clos-Func using this well-formedness premise, we obtain $\Psi \mid \Theta \vdash^{\gamma_2} v : A\ \mathbf{val}$.

Case VT-Clos-Rec**:** This case is analogous to the previous one.

Case VT-Code**:** By inversion of VT-Code, we have $v = \langle e^0 \rangle$ and $A = \forall \varrho_1 \ldots \forall \varrho_k. \langle A_1 \rangle^{\delta'}$. Moreover, the derivation gives

$$\Psi \mid^{\gamma_1} \varepsilon \vdash_{\mathbf{id}}^{\gamma_1} \langle e^0 \rangle : \forall \varrho_1 \ldots \forall \varrho_k. \langle A_1 \rangle^{\delta'}$$

together with $\Psi \vdash \Theta\ \mathbf{wf}$. By inverting this typing derivation, for some classifiers δ_0 to δ_k with $\delta_0 = \gamma_1$, we have

$$\Psi \mid^{\delta_0} [\varrho_1]^{\delta_1}, \ldots, [\varrho_k]^{\delta_k}, \blacktriangleright^{\delta'} \vdash_{\mathbf{id}}^{\delta_k, \delta'} e^0 : A_1$$

and the side conditions on the classifier declarations, namely $\delta_i \notin \mathbf{FC}(\varrho_{i+1}) \cup \cdots \cup \mathbf{FC}(\varrho_k) \cup \mathbf{FC}(A_1) \cup \{\delta'\}$ for $1 \le i \le k$. From the assumption $\Psi \vdash^{\gamma_2} \forall \varrho_1 \ldots \forall \varrho_k. \langle A_1 \rangle^{\delta'}\ \mathbf{wf}$, inversion gives $\Psi \mid^{\gamma_2} [\varrho_1]^{\delta_1}, \ldots, [\varrho_k]^{\delta_k}, \blacktriangleright^{\delta'} \vdash^{\delta_k[\delta_0 := \gamma_2], \delta'} A_1\ \mathbf{wf}$. Hence, by Lemma 26,

$$\Psi \mid^{\gamma_2} [\varrho_1]^{\delta_1}, \ldots, [\varrho_k]^{\delta_k}, \blacktriangleright^{\delta'} \vdash_{\mathbf{id}}^{\delta_k[\delta_0 := \gamma_2], \delta'} e^0 : A_1.$$

By reapplying R-T-Quote and R-T-Polycls, we obtain

$$\Psi \mid^{\gamma_2} \varepsilon \vdash_{\mathbf{id}}^{\gamma_2} \langle e^0 \rangle : \forall \varrho_1 \ldots \forall \varrho_k. \langle A_1 \rangle^{\delta'}.$$

Therefore, by reapplying VT-Code, we obtain $\Psi \mid \Theta \vdash^{\gamma_2} v : A\ \mathbf{val}$.

Case VT-Loc**:** By inversion of VT-Loc, we have $v = l$ and $A = A_1\ \mathbf{ref}$, together with $\Psi \vdash \Theta\ \mathbf{wf}$, $\mathbf{lookupStore}(\Theta, l) = \mathbf{Done}\,(A_1)$, and $\Psi \vdash^{\gamma_1} A_1\ \mathbf{wf}$. From the assumption $\Psi \vdash^{\gamma_2} A_1\ \mathbf{ref}\ \mathbf{wf}$, inversion of R-WF-T-Ref gives $\Psi \vdash^{\gamma_2} A_1\ \mathbf{wf}$. Therefore, by reapplying VT-Loc, we obtain $\Psi \mid \Theta \vdash^{\gamma_2} l : A_1\ \mathbf{ref}\ \mathbf{val}$.

□

Lemma 28 (Classifier Application for Values)**.** *Suppose that*

1. $\Psi_1 \mid \Theta \vdash^{\delta_0} v : \forall \gamma_1 :\succeq \gamma'_1, \ldots, \gamma_{k_1} :\succeq \gamma'_{k_1}.A\ \mathbf{val}$,
2. $\Psi_1 \vdash \gamma'_i \preceq \delta_i$ *for each* i *with* $1 \leq i \leq k_1$, *and*
3. $\Psi_1 \vdash \delta_{i+1} \subseteq \delta_i$ *for each* i *with* $0 \leq i < k_1$.

Then there exists Ψ_2 *such that* $\Psi_1 \subseteq \Psi_2$, $\mathbf{Dom}_V(\Psi_2) = \mathbf{Dom}_V(\Psi_1)$, *and*

$$\Psi_2 \mid \Theta \vdash^{\delta_0} v : A[\gamma_1 \coloneqq \delta_1, \ldots, \gamma_{k_1} \coloneqq \delta_{k_1}]\ \mathbf{val}.$$

Proof. We proceed by inversion on $\Psi_1 \mid \Theta \vdash^{\delta_0} v : \forall \gamma_1 :\succeq \gamma'_1, \ldots, \gamma_{k_1} :\succeq \gamma'_{k_1}.A\ \mathbf{val}$. The only possible cases are the closure cases and the code case. In each case, we instantiate the typing premise for the underlying constructor by R-T-Appcls, and then reapply the corresponding value typing rule.

Case VT-Clos-Func**:** By inversion, we have $v = \mathbf{clos}\,(E, R, \lambda x.\ e^0)$ and $A = \forall \varrho_1 \ldots \forall \varrho_{k_2}.A_1 \to A_2$. Moreover, the derivation gives

$$\Psi_1 \mid^{\gamma'_0} \varepsilon \vdash^{\gamma'_0}_R \lambda x.\ e^0 : \forall \gamma_1 :\succeq \gamma'_1, \ldots, \gamma_{k_1} :\succeq \gamma'_{k_1}.A,$$

together with $\Psi_1 \mid \Theta \vdash^{\gamma'_0} E\ \mathbf{wf}$.

Choose a fresh classifier δ'' and let $\Psi_2 = \Psi_1, \delta''(:\succeq \gamma'_0 / :\supseteq \delta_1)$. Then $\Psi_1 \subseteq \Psi_2$ and $\mathbf{Dom}_V(\Psi_2) = \mathbf{Dom}_V(\Psi_1)$. Moreover, by Lemma 11 we obtain

$$\Psi_2 \mid^{\gamma'_0} \varepsilon \vdash^{\gamma'_0}_R \lambda x.\ e^0 : \forall \gamma_1 :\succeq \gamma'_1, \ldots, \gamma_{k_1} :\succeq \gamma'_{k_1}.A.$$

Since $\Psi_2 \vdash \gamma'_0 \preceq \delta''$, Lemma 13 gives

$$\Psi_2 \mid^{\delta''} \varepsilon \vdash^{\delta''}_R \lambda x.\ e^0 : \forall \gamma_1 :\succeq \gamma'_1, \ldots, \gamma_{k_1} :\succeq \gamma'_{k_1}.A.$$

The classifier sequence $\delta'', \delta_1, \ldots, \delta_{k_1}$ satisfies the premises of R-T-Appcls: the first reachability premise follows from the declaration of Ψ_2, and the remaining premises follow by Lemma 11 from the assumptions of the lemma. Hence, writing $\sigma = \gamma_1 \coloneqq \delta_1, \ldots, \gamma_{k_1} \coloneqq \delta_{k_1}$, we obtain

$$\Psi_2 \mid^{\delta''} \varepsilon \vdash^{\delta''}_R \lambda x.\ e^0 : A[\sigma].$$

Moreover, $\Psi_2 \mid \Theta \vdash^{\delta''} E\ \mathbf{wf}$ follows by WF-VEnv-Shut.

By inversion of the well-formedness premise in the original value typing derivation, we have

$$\Psi_1 \mid^{\gamma'_0} [\gamma_1 :\succeq \gamma'_1, \ldots, \gamma_{k_1} :\succeq \gamma'_{k_1}]^{\gamma_0} \vdash^{\gamma_0} A\ \mathbf{wf}.$$

Weakening this judgment to Ψ_2 and applying Lemma 16 yields $\Psi_2 \vdash^{\delta_0} A[\sigma]\ \mathbf{wf}$.

Therefore, by reapplying VT-Clos-Func, we obtain $\Psi_2 \mid \Theta \vdash^{\delta_0} v : A[\sigma]\ \mathbf{val}$.

Case VT-Clos-Rec**:** This case is analogous to the previous one, using VT-Clos-Rec instead of VT-Clos-Func.

Case VT-Code**:** By inversion, we have $v = \langle e^0 \rangle$ and $A = \forall \varrho_1 \ldots \forall \varrho_{k_2}.\langle A_1 \rangle^{\delta'}$. The derivation gives

$$\Psi_1 \mid^{\delta_0} \varepsilon \vdash^{\delta_0}_{\mathbf{id}} \langle e^0 \rangle : \forall \gamma_1 :\succeq \gamma'_1, \ldots, \gamma_{k_1} :\succeq \gamma'_{k_1}.A$$

together with $\Psi_1 \vdash \Theta\ \mathbf{wf}$. By R-T-Appcls and the assumptions of the lemma, we obtain

$$\Psi_1 \mid^{\delta_0} \varepsilon \vdash^{\delta_0}_{\mathbf{id}} \langle e^0 \rangle : A[\gamma_1 \coloneqq \delta_1, \ldots, \gamma_{k_1} \coloneqq \delta_{k_1}].$$

Let $\Psi_2 = \Psi_1$. Then $\Psi_1 \subseteq \Psi_2$ and $\mathbf{Dom}_V(\Psi_2) = \mathbf{Dom}_V(\Psi_1)$. Therefore, by reapplying VT-Code, we obtain $\Psi_2 \mid \Theta \vdash^{\delta_0} v : A[\gamma_1 \coloneqq \delta_1, \ldots, \gamma_{k_1} \coloneqq \delta_{k_1}]\ \mathbf{val}$.

□

3.7 Promotion Lemma

Lemma 29 (Promotion for $\blacktriangleright$)**.** *1. If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \blacktriangleright^{\gamma_2}, \Gamma \vdash^{\overrightarrow{\gamma_3}^+}$ **wf**, *then* $\Psi \mid^{\overrightarrow{\gamma_1}^+, \gamma_2} \Gamma \vdash^{\overrightarrow{\gamma_3}^+}$ **wf**.

2. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \blacktriangleright^{\gamma_2}, \Gamma \vdash^{\overrightarrow{\gamma_3}^+} A$ **wf**, *then* $\Psi \mid^{\overrightarrow{\gamma_1}^+, \gamma_2} \Gamma \vdash^{\overrightarrow{\gamma_3}^+} A$ **wf**.

3. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \blacktriangleright^{\gamma_2}, \Gamma \vdash \gamma_3 \preceq \gamma_4$, *then* $\Psi \mid^{\overrightarrow{\gamma_1}^+, \gamma_2} \Gamma \vdash \gamma_3 \preceq \gamma_4$.

4. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \blacktriangleright^{\gamma_2}, \Gamma \vdash \gamma_3 \subseteq \gamma_4$, *then* $\Psi \mid^{\overrightarrow{\gamma_1}^+, \gamma_2} \Gamma \vdash \gamma_3 \subseteq \gamma_4$.

5. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \blacktriangleright^{\gamma_2}, \Gamma \vdash_R^{\overrightarrow{\gamma_3}^+} e : A$, *then* $\Psi \mid^{\overrightarrow{\gamma_1}^+, \gamma_2} \Gamma \vdash_R^{\overrightarrow{\gamma_3}^+} e : A$.

Proof. By mutual induction on the given derivation. □

Lemma 30 (Promotion for $\blacktriangleleft$)**.** *1. If* $\Psi \mid^{\overrightarrow{\gamma_1}, \delta_0, \ldots, \delta_k} \blacktriangleleft_k^{\gamma_2}, \Gamma \vdash^{\overrightarrow{\gamma_3}^+}$ **wf**, *then* $\Psi, \gamma_2(:\succeq \delta_0 / :\supseteq \delta_k) \mid^{\overrightarrow{\gamma_1}, \gamma_2} \Gamma \vdash^{\overrightarrow{\gamma_3}^+}$ **wf**.

2. *If* $\Psi \mid^{\overrightarrow{\gamma_1}, \delta_0, \ldots, \delta_k} \blacktriangleleft_k^{\gamma_2}, \Gamma \vdash^{\overrightarrow{\gamma_3}^+} A$ **wf**, *then* $\Psi, \gamma_2(:\succeq \delta_0 / :\supseteq \delta_k) \mid^{\overrightarrow{\gamma_1}, \gamma_2} \Gamma \vdash^{\overrightarrow{\gamma_3}^+} A$ **wf**.

3. *If* $\Psi \mid^{\overrightarrow{\gamma_1}, \delta_0, \ldots, \delta_k} \blacktriangleleft_k^{\gamma_2}, \Gamma \vdash \gamma_3 \preceq \gamma_4$, *then* $\Psi, \gamma_2(:\succeq \delta_0 / :\supseteq \delta_k) \mid^{\overrightarrow{\gamma_1}, \gamma_2} \Gamma \vdash \gamma_3 \preceq \gamma_4$.

4. *If* $\Psi \mid^{\overrightarrow{\gamma_1}, \delta_0, \ldots, \delta_k} \blacktriangleleft_k^{\gamma_2}, \Gamma \vdash \gamma_3 \subseteq \gamma_4$, *then* $\Psi, \gamma_2(:\succeq \delta_0 / :\supseteq \delta_k) \mid^{\overrightarrow{\gamma_1}, \gamma_2} \Gamma \vdash \gamma_3 \subseteq \gamma_4$.

5. *If* $\Psi \mid^{\overrightarrow{\gamma_1}, \delta_0, \ldots, \delta_k} \blacktriangleleft_k^{\gamma_2}, \Gamma \vdash_R^{\overrightarrow{\gamma_3}^+} e : A$, *then* $\Psi, \gamma_2(:\succeq \delta_0 / :\supseteq \delta_k) \mid^{\overrightarrow{\gamma_1}, \gamma_2} \Gamma \vdash_R^{\overrightarrow{\gamma_3}^+} e : A$.

Proof. By mutual induction on the given derivation. □

Lemma 31 (Promotion for Variables)**.** *Suppose* $y \notin \mathbf{Dom}_V(\Psi)$.

1. *If* $\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} x{:}^{\gamma_3} A_1, \Gamma \vdash^{\overrightarrow{\gamma_4}^+}$ **wf**, *then* $\Psi, y : A_1 \mathbin{@} \gamma_3 :\succeq \gamma_2 \mid^{\overrightarrow{\gamma_1}, \gamma_3} \Gamma \vdash^{\overrightarrow{\gamma_4}^+}$ **wf**.

2. *If* $\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} x{:}^{\gamma_3} A_1, \Gamma \vdash^{\overrightarrow{\gamma_4}^+} A_2$ **wf**, *then* $\Psi, y : A_1 \mathbin{@} \gamma_3 :\succeq \gamma_2 \mid^{\overrightarrow{\gamma_1}, \gamma_3} \Gamma \vdash^{\overrightarrow{\gamma_4}^+} A_2$ **wf**.

3. *If* $\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} x{:}^{\gamma_3} A_1, \Gamma \vdash \gamma_4 \preceq \gamma_5$, *then* $\Psi, y : A_1 \mathbin{@} \gamma_3 :\succeq \gamma_2 \mid^{\overrightarrow{\gamma_1}, \gamma_3} \Gamma \vdash \gamma_4 \preceq \gamma_5$.

4. *If* $\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} x{:}^{\gamma_3} A_1, \Gamma \vdash \gamma_4 \subseteq \gamma_5$, *then* $\Psi, y : A_1 \mathbin{@} \gamma_3 :\succeq \gamma_2 \mid^{\overrightarrow{\gamma_1}, \gamma_3} \Gamma \vdash \gamma_4 \subseteq \gamma_5$.

5. *If* $\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} x{:}^{\gamma_3} A_1, \Gamma \vdash_R^{\overrightarrow{\gamma_4}^+} e : A_2$, *then* $\Psi, y : A_1 \mathbin{@} \gamma_3 :\succeq \gamma_2 \mid^{\overrightarrow{\gamma_1}, \gamma_3} \Gamma \vdash_{R, x:=y}^{\overrightarrow{\gamma_4}^+} e : A_2$.

Proof. By mutual induction on the given derivation. □

Lemma 32 (Promotion for Polymorphic Classifier Declarations)**.** *1. If* $\Psi \mid^{\overrightarrow{\delta_1}, \gamma'_0} [\gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k]^{\gamma_0}, \Gamma \vdash^{\overrightarrow{\delta_2}^+}$ **wf**, *then* $\Psi, [\gamma_0 :\succeq \gamma'_0, \ldots, \gamma_k :\succeq \gamma'_k] \mid^{\overrightarrow{\delta_1}, \gamma_0} \Gamma \vdash^{\overrightarrow{\delta_2}^+}$ **wf**.

2. *If* $\Psi \mid^{\overrightarrow{\delta_1}, \gamma'_0} [\gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k]^{\gamma_0}, \Gamma \vdash^{\overrightarrow{\delta_2}^+} A$ **wf**, *then* $\Psi, [\gamma_0 :\succeq \gamma'_0, \ldots, \gamma_k :\succeq \gamma'_k] \mid^{\overrightarrow{\delta_1}, \gamma_0} \Gamma \vdash^{\overrightarrow{\delta_2}^+} A$ **wf**.

3. *If* $\Psi \mid^{\overrightarrow{\delta_1}, \gamma'_0} [\gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k]^{\gamma_0}, \Gamma \vdash \delta_3 \preceq \delta_4$, *then* $\Psi, [\gamma_0 :\succeq \gamma'_0, \ldots, \gamma_k :\succeq \gamma'_k] \mid^{\overrightarrow{\delta_1}, \gamma_0} \Gamma \vdash \delta_3 \preceq \delta_4$.

4. *If* $\Psi \mid^{\overrightarrow{\delta_1}, \gamma'_0} [\gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k]^{\gamma_0}, \Gamma \vdash \delta_3 \subseteq \delta_4$, *then* $\Psi, [\gamma_0 :\succeq \gamma'_0, \ldots, \gamma_k :\succeq \gamma'_k] \mid^{\overrightarrow{\delta_1}, \gamma_0} \Gamma \vdash \delta_3 \subseteq \delta_4$.

5. *If* $\Psi \mid^{\overrightarrow{\delta_1}, \gamma'_0} [\gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k]^{\gamma_0}, \Gamma \vdash_R^{\overrightarrow{\delta_2}^+} e : A$, *then* $\Psi, [\gamma_0 :\succeq \gamma'_0, \ldots, \gamma_k :\succeq \gamma'_k] \mid^{\overrightarrow{\delta_1}, \gamma_0} \Gamma \vdash_R^{\overrightarrow{\delta_2}^+} e : A$.

Proof. By mutual induction on the given derivation. □

Corollary 3 (Promotion)**.** *1. If* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \blacktriangleright^{\gamma_2} \vdash_R^{\overrightarrow{\gamma_1}^+, \gamma_2} e : A$, *then* $\Psi \mid^{\overrightarrow{\gamma_1}^+, \gamma_2} \varepsilon \vdash_R^{\overrightarrow{\gamma_1}^+, \gamma_2} e : A$.

2. *If* $\Psi \mid^{\overrightarrow{\gamma_1}, \delta_0, \ldots, \delta_k} \blacktriangleleft_k^{\gamma_2} \vdash_R^{\overrightarrow{\gamma_1}, \gamma_2} e : A$, *then* $\Psi, \gamma_2(:\succeq \delta_0 / :\supseteq \delta_k) \mid^{\overrightarrow{\gamma_1}, \gamma_2} \varepsilon \vdash_R^{\overrightarrow{\gamma_1}, \gamma_2} e : A$.

3. *If* $\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} x{:}^{\gamma_3} A_1 \vdash_R^{\overrightarrow{\gamma_1}, \gamma_3} e : A_2$ *and* $y \notin \mathbf{Dom}_V(\Psi)$, *then* $\Psi, y : A_1 \mathbin{@} \gamma_3 :\succeq \gamma_2 \mid^{\overrightarrow{\gamma_1}, \gamma_3} \varepsilon \vdash_{R, x:=y}^{\overrightarrow{\gamma_1}, \gamma_3} e : A_2$.

4. *If* $\Psi \mid^{\overrightarrow{\delta_1}, \gamma'_0} [\gamma_1 :\succeq \gamma'_1, \ldots, \gamma_k :\succeq \gamma'_k]^{\gamma_0} \vdash_R^{\overrightarrow{\delta_1}, \gamma_0} e : A$, *then* $\Psi, [\gamma_0 :\succeq \gamma'_0, \ldots, \gamma_k :\succeq \gamma'_k] \mid^{\overrightarrow{\delta_1}, \gamma_0} \varepsilon \vdash_R^{\overrightarrow{\delta_1}, \gamma_0} e : A$.

3.8 Demotion Lemma

Definition 12 (Disjointness of Classifiers)**.** *We write* $\Psi \vdash \overrightarrow{\gamma_1} \# \overrightarrow{\gamma_2}$ *if* $\Psi \vdash \gamma_2' \subseteq \gamma_1'$ *does not hold for any* $\gamma_1' \in \{\overrightarrow{\gamma_1}\}$ *and* $\gamma_2' \in \{\overrightarrow{\gamma_2}\}$*.*

Lemma 33 (Demotion for Variables)**.** *Suppose* $x : A_1 \mathbin{@} \gamma_2 :\succeq \gamma_4 \in \Psi$*,* $\Psi \vdash \overrightarrow{\gamma_1} \# \gamma_2$*, and* $\gamma_5 \notin \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma)$*.*

1. *If* $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_2}\, \Gamma \vdash^{\overrightarrow{\gamma_3}^+} \mathbf{wf}$*, then* $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_4}\, x{:}^{\gamma_5} A_1, \Gamma[\gamma_2 \coloneqq \gamma_5] \vdash^{\overrightarrow{\gamma_3}^+[\gamma_2 \coloneqq \gamma_5]} \mathbf{wf}$*.*
2. *If* $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_2}\, \Gamma \vdash^{\overrightarrow{\gamma_3}^+} A_2\ \mathbf{wf}$*, then* $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_4}\, x{:}^{\gamma_5} A_1, \Gamma[\gamma_2 \coloneqq \gamma_5] \vdash^{\overrightarrow{\gamma_3}^+[\gamma_2 \coloneqq \gamma_5]} A_2[\gamma_2 \coloneqq \gamma_5]\ \mathbf{wf}$*.*
3. *If* $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_2}\, \Gamma \vdash \gamma_6 \preceq \gamma_7$ *and either* $\gamma_7 \in \mathbf{Dom}_C(\Gamma)$ *or there exists* $\gamma_8 \in \{\overrightarrow{\gamma_1}, \gamma_2\}$ *such that* $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_2}\, \Gamma \vdash \gamma_7 \subseteq \gamma_8$*, then* $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_4}\, x{:}^{\gamma_5} A_1, \Gamma[\gamma_2 \coloneqq \gamma_5] \vdash \gamma_6[\gamma_2 \coloneqq \gamma_5] \preceq \gamma_7[\gamma_2 \coloneqq \gamma_5]$*.*
4. *If* $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_2}\, \Gamma \vdash \gamma_6 \subseteq \gamma_7$ *and either* $\gamma_7 \in \mathbf{Dom}_C(\Gamma)$ *or there exists* $\gamma_8 \in \{\overrightarrow{\gamma_1}, \gamma_2\}$ *such that* $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_2}\, \Gamma \vdash \gamma_7 \subseteq \gamma_8$*, then* $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_4}\, x{:}^{\gamma_5} A_1, \Gamma[\gamma_2 \coloneqq \gamma_5] \vdash \gamma_6[\gamma_2 \coloneqq \gamma_5] \subseteq \gamma_7[\gamma_2 \coloneqq \gamma_5]$*.*
5. *If* $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_2}\, \Gamma \vdash^{\overrightarrow{\gamma_3}^+}_{\mathbf{id}} e : A_2$*, then* $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_4}\, x{:}^{\gamma_5} A_1, \Gamma[\gamma_2 \coloneqq \gamma_5] \vdash^{\overrightarrow{\gamma_3}^+[\gamma_2 \coloneqq \gamma_5]}_{\mathbf{id}} e : A_2[\gamma_2 \coloneqq \gamma_5]$*.*

Proof. We proceed by mutual induction on the given derivation. We only discuss the cases for visibility judgments and expression typing.

Case Item 1**:** By induction on the context well-formedness derivation. We spell out the case for R-WF-Ctx-▶; the other cases are immediate by the induction hypothesis and reapplication of the same well-formedness rule.

Suppose the last rule is R-WF-Ctx-▶. By inversion, $\Gamma = \Gamma_1, \blacktriangleright^{\delta_1}$ and $\overrightarrow{\gamma_3}^+ = \overrightarrow{\delta_2}, \delta_3, \delta_1$, with premises

$$\Psi \,|^{\overrightarrow{\gamma_1},\gamma_2}\, \Gamma_1 \vdash^{\overrightarrow{\delta_2},\delta_3} \mathbf{wf} \quad \text{and} \quad \Psi \,|^{\overrightarrow{\gamma_1},\gamma_2}\, \Gamma_1 \vdash \delta_1 \subseteq \delta_3.$$

The induction hypothesis gives the demoted well-formedness of Γ_1. Moreover, by Lemma 22, $\delta_3 \in \{\overrightarrow{\gamma_1}, \gamma_2\} \cup \mathbf{Dom}_C(\Gamma_1) \cup \mathbf{FC}(\Gamma_1)$, and hence the side condition of Item 4 holds for the visibility premise. Thus

$$\Psi \,|^{\overrightarrow{\gamma_1},\gamma_4}\, x{:}^{\gamma_5} A_1[\gamma_2 \coloneqq \gamma_5], \Gamma_1[\gamma_2 \coloneqq \gamma_5] \vdash \delta_1[\gamma_2 \coloneqq \gamma_5] \subseteq \delta_3[\gamma_2 \coloneqq \gamma_5].$$

Reapplying R-WF-Ctx-▶ yields the desired well-formedness judgment.

Case Item 2**:** By induction on the type well-formedness derivation.

Case Item 3**:** This case is analogous to Item 4, using the inversion lemma and rules for $\preceq$ in place of those for $\subseteq$.

Case Item 4**:** By induction with respect to $\mathbf{deg}(\Psi, \Gamma, \gamma_7)$.

By the side condition of the lemma, either there exists $\gamma_8 \in \{\overrightarrow{\gamma_1}, \gamma_2\}$ such that $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_2}\, \Gamma \vdash \gamma_7 \subseteq \gamma_8$, or $\gamma_7 \in \mathbf{Dom}_C(\Gamma)$. We first consider the case where $\gamma_7 \notin \mathbf{Dom}_C(\Gamma)$. Then, we have $\gamma_8 \in \{\overrightarrow{\gamma_1}, \gamma_2\}$ such that $\Psi \,|^{\overrightarrow{\gamma_1},\gamma_2}\, \Gamma \vdash \gamma_7 \subseteq \gamma_8$. By disjointness of $\mathbf{Dom}_C(\Gamma)$ and $\mathbf{Dom}_C(\Psi)$, we have $\gamma_7 \in \mathbf{Dom}_C(\Psi)$. Therefore $\Psi \vdash \gamma_6 \subseteq \gamma_7$ by Lemma 5. Moreover, Lemma 1 and Item 1 give

$$\Psi \,|^{\overrightarrow{\gamma_1},\gamma_4}\, x{:}^{\gamma_5} A_1[\gamma_2 \coloneqq \gamma_5], \Gamma[\gamma_2 \coloneqq \gamma_5] \vdash^{\cdots} \mathbf{wf}.$$

Therefore, by Lemma 6,

$$\Psi \,|^{\overrightarrow{\gamma_1},\gamma_4}\, x{:}^{\gamma_5} A_1[\gamma_2 \coloneqq \gamma_5], \Gamma[\gamma_2 \coloneqq \gamma_5] \vdash \gamma_6 \subseteq \gamma_7.$$

We conclude by cases on whether the substitution $\gamma_2 \coloneqq \gamma_5$ affects γ_6 or γ_7.

Case $\gamma_2 \neq \gamma_6$ and $\gamma_2 \neq \gamma_7$**:** The substitution leaves both endpoints unchanged, so the required judgment follows immediately.

Case $\gamma_2 \neq \gamma_6$ and $\gamma_2 = \gamma_7$**:** By Lemma 9 and $x : A_1 \mathbin{@} \gamma_2 :\succeq \gamma_4 \in \Psi$, we have $\Psi \vdash \gamma_6 \subseteq \gamma_4$. By Lemma 6,

$$\Psi \,|^{\overrightarrow{\gamma_1},\gamma_4}\, x{:}^{\gamma_5} A_1[\gamma_2 \coloneqq \gamma_5], \Gamma[\gamma_2 \coloneqq \gamma_5] \vdash \gamma_6[\gamma_2 \coloneqq \gamma_5] \subseteq \gamma_4.$$

Reapplying R-$\preceq$-Var in the demoted context and then R-$\subseteq$-Lift gives

$$\Psi \,|^{\overrightarrow{\gamma_1},\gamma_4}\, x{:}^{\gamma_5} A_1[\gamma_2 \coloneqq \gamma_5], \Gamma[\gamma_2 \coloneqq \gamma_5] \vdash \gamma_4 \subseteq \gamma_5.$$

The conclusion follows by R-$\subseteq$-Trans.

Case $\gamma_2 = \gamma_6 = \gamma_7$**:** Then $\gamma_6[\gamma_2 \coloneqq \gamma_5] = \gamma_7[\gamma_2 \coloneqq \gamma_5] = \gamma_5$. The required judgment follows from R-$\preceq$-Refl and R-$\subseteq$-Lift.

Case $\gamma_2 = \gamma_6$ and $\gamma_2 \neq \gamma_7$**:** Since $\gamma_6 = \gamma_2$, we have $\Psi \vdash \gamma_2 \subseteq \gamma_7$. Moreover, for some $\gamma_8 \in \{\overrightarrow{\gamma_1}, \gamma_2\}$ we have $\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} \Gamma \vdash \gamma_7 \subseteq \gamma_8$. By Lemma 5, this gives $\Psi \vdash \gamma_7 \subseteq \gamma_8$; together with $\Psi \vdash \gamma_2 \subseteq \gamma_7$ and R-$\subseteq$-Trans, we get $\Psi \vdash \gamma_2 \subseteq \gamma_8$. Since $\Psi \vdash \overrightarrow{\gamma_1} \# \gamma_2$, the only possible such γ_8 is γ_2. Hence $\Psi \vdash \gamma_7 \subseteq \gamma_2$. Applying Lemma 4 to $\Psi \vdash \gamma_2 \subseteq \gamma_7$ and $\Psi \vdash \gamma_7 \subseteq \gamma_2$ yields $\gamma_2 = \gamma_7$, contradicting the case assumption. Thus this case is impossible.

It remains to consider the case $\gamma_7 \in \mathbf{Dom}_C(\Gamma)$. If $\gamma_6 = \gamma_7$, then $\gamma_7[\gamma_2 \coloneqq \gamma_5]$ is declared in the demoted context by Item 1. Hence the desired judgment follows from R-$\preceq$-Refl and R-$\subseteq$-Lift. Thus assume $\gamma_6 \neq \gamma_7$. We perform case analysis using Lemma 10. In each case, the inversion gives a derivation $\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} \Gamma \vdash \gamma_6 \subseteq \delta_3$ where $\mathbf{deg}(\Psi, \Gamma, \delta_3) < \mathbf{deg}(\Psi, \Gamma, \gamma_7)$ and $\delta_3 \in \{\overrightarrow{\gamma_1}, \gamma_2\} \cup \mathbf{Dom}_C(\Gamma) \cup \mathbf{FC}(\Gamma)$. The membership condition follows from Lemma 22 when δ_3 comes from the output stack of the corresponding context prefix. In the remaining polymorphic-classifier declaration cases, δ_3 appears as the right-hand side of a declaration $\gamma_7 :\succeq \delta_3$, so the condition follows from the definitions of $\mathbf{Dom}_C(\Gamma)$ and $\mathbf{FC}(\Gamma)$. Hence, the induction hypothesis gives

$$\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_4}\ x{:}^{\gamma_5} A_1[\gamma_2 \coloneqq \gamma_5], \Gamma[\gamma_2 \coloneqq \gamma_5] \vdash \gamma_6[\gamma_2 \coloneqq \gamma_5] \subseteq \delta_3[\gamma_2 \coloneqq \gamma_5].$$

Reapplying the corresponding context rule gives

$$\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_4}\ x{:}^{\gamma_5} A_1[\gamma_2 \coloneqq \gamma_5], \Gamma[\gamma_2 \coloneqq \gamma_5] \vdash \delta_3[\gamma_2 \coloneqq \gamma_5] \subseteq \gamma_7[\gamma_2 \coloneqq \gamma_5].$$

The desired judgment follows by R-$\subseteq$-Trans.

Case Item 5**:** By induction on the typing derivation. We spell out R-T-Var, R-T-Name, and R-T-Splice; the remaining cases are immediate or analogous by the induction hypothesis.

Case R-T-Var**:** By inversion, the expression is a variable z and the derivation gives $\mathbf{lookup}\,(\Gamma, z) = \mathbf{Done}\,(A_2, \delta_1)$,

$$\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} \Gamma \vdash^{\overrightarrow{\delta_2}, \delta_3} \mathbf{wf} \quad \text{and} \quad \Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} \Gamma \vdash \delta_1 \preceq \delta_3.$$

By Item 1, the well-formed context premise is preserved by demotion. It remains to check the side condition needed to apply Item 3 to the reachability premise. By Lemma 22,

$$\delta_3 \in \{\overrightarrow{\gamma_1}, \gamma_2\} \cup \mathbf{Dom}_C(\Gamma) \cup \mathbf{FC}(\Gamma).$$

If $\delta_3 \in \mathbf{Dom}_C(\Gamma)$, the side condition is immediate. If $\delta_3 \in \{\overrightarrow{\gamma_1}, \gamma_2\}$, it is also immediate by taking $\gamma_3 = \delta_3$ and using reflexivity. Finally, if $\delta_3 \in \mathbf{FC}(\Gamma)$, then Corollary 2 gives some $\gamma_3 \in \{\overrightarrow{\gamma_1}, \gamma_2\}$ such that $\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} \Gamma \vdash \delta_3 \subseteq \gamma_3$. Thus the side condition holds in all cases, and Item 3 gives the demoted reachability premise. The lookup premise is preserved by classifier substitution. Reapplying R-T-Var gives the required typing judgment.

Case R-T-Name**:** By inversion, the expression is a variable z, $\mathbf{lookup}\,(\Gamma, z) = \mathbf{Fail}$, and since the renaming environment is $\mathbf{id}$, the reserved-name premise is $z : A_2 \mathbin{@} \delta_1 :\succeq \delta_2 \in \Psi$. If $z = x$, then this premise is the distinguished declaration $x : A_1 \mathbin{@} \gamma_2 :\succeq \gamma_4 \in \Psi$. Then we have

$$\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} \Gamma \vdash^{\overrightarrow{\delta_2}, \delta_3} \mathbf{wf} \quad \text{and} \quad \Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} \Gamma \vdash \gamma_2 \preceq \delta_3.$$

We first check the side condition needed to apply Item 3. By Lemma 22,

$$\delta_3 \in \{\overrightarrow{\gamma_1}, \gamma_2\} \cup \mathbf{Dom}_C(\Gamma) \cup \mathbf{FC}(\Gamma).$$

If $\delta_3 \in \mathbf{Dom}_C(\Gamma)$, the side condition is immediate. If $\delta_3 \in \{\overrightarrow{\gamma_1}, \gamma_2\}$, it is also immediate by taking $\gamma_3 = \delta_3$ and using reflexivity. Finally, if $\delta_3 \in \mathbf{FC}(\Gamma)$, then Corollary 2 gives some $\gamma_3 \in \{\overrightarrow{\gamma_1}, \gamma_2\}$ such that $\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} \Gamma \vdash \delta_3 \subseteq \gamma_3$. Thus the side condition holds in all cases, and Item 3 gives

$$\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_4}\ x{:}^{\gamma_5} A_1[\gamma_2 \coloneqq \gamma_5], \Gamma[\gamma_2 \coloneqq \gamma_5] \vdash \gamma_5 \preceq \delta_3[\gamma_2 \coloneqq \gamma_5].$$

Moreover, Item 1 gives the demoted context well-formedness premise, and the demoted context contains the ordinary binding $x{:}^{\gamma_5} A_1[\gamma_2 := \gamma_5]$. Reapplying R-T-Var yields the desired judgment.

If $z \neq x$, the same reserved-name declaration remains available in Ψ. The context well-formedness premise is preserved by Item 1. The reachability premise is preserved by Item 3, using Lemma 22 for the side condition on the output classifier. Reapplying R-T-Name yields the desired judgment.

Case R-T-Splice**:** By inversion, the derivation gives premises

$$\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash^{\overrightarrow{\delta_1},\delta_0,\ldots,\delta_k} \mathbf{wf},$$

$$\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma, \blacktriangleleft_k^{\delta_4} \vdash_{\mathbf{id}}^{\overrightarrow{\delta_1},\delta_4} e : \langle A_3 \rangle^{\delta_5}$$

and $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash \delta_5 \preceq \delta_k$. By Item 1, the context well-formedness premise is preserved. Applying the induction hypothesis to the body typing premise gives the demoted body typing judgment under $\Gamma[\gamma_2 := \gamma_5], \blacktriangleleft_k^{\delta_4}$. Finally, we check the side condition needed to demote the reachability premise $\delta_5 \preceq \delta_k$. By Lemma 22,

$$\delta_k \in \{\overrightarrow{\gamma_1}, \gamma_2\} \cup \mathbf{Dom}_C(\Gamma) \cup \mathbf{FC}(\Gamma).$$

If $\delta_k \in \mathbf{Dom}_C(\Gamma)$, the side condition is immediate. If $\delta_k \in \{\overrightarrow{\gamma_1}, \gamma_2\}$, it is also immediate by taking $\gamma_3 = \delta_k$ and using reflexivity. Finally, if $\delta_k \in \mathbf{FC}(\Gamma)$, then Corollary 2 gives some $\gamma_3 \in \{\overrightarrow{\gamma_1}, \gamma_2\}$ such that $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash \delta_k \subseteq \gamma_3$. Thus the side condition holds in all cases, and Item 3 preserves the reachability premise $\delta_5 \preceq \delta_k$. Reapplying R-T-Splice gives the desired typing judgment.

□

Lemma 34 (Demotion for $\blacktriangleright$)**.** *Suppose* $\Psi \vdash \gamma_3 \subseteq \gamma_2$*.*

1. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+,\gamma_2,\gamma_3} \Gamma \vdash^{\overrightarrow{\gamma_4}^+} \mathbf{wf}$*, then* $\Psi \mid^{\overrightarrow{\gamma_1}^+,\gamma_2} \blacktriangleright^{\gamma_3}, \Gamma \vdash^{\overrightarrow{\gamma_4}^+} \mathbf{wf}$*.*
2. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+,\gamma_2,\gamma_3} \Gamma \vdash^{\overrightarrow{\gamma_4}^+} A\ \mathbf{wf}$*, then* $\Psi \mid^{\overrightarrow{\gamma_1}^+,\gamma_2} \blacktriangleright^{\gamma_3}, \Gamma \vdash^{\overrightarrow{\gamma_4}^+} A\ \mathbf{wf}$*.*
3. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+,\gamma_2,\gamma_3} \Gamma \vdash \gamma_4 \preceq \gamma_5$ *and either* $\gamma_5 \in \mathbf{Dom}_C(\Gamma)$ *or* $\Psi \vdash \gamma_5 \subseteq \gamma_3$*, then* $\Psi \mid^{\overrightarrow{\gamma_1}^+,\gamma_2} \blacktriangleright^{\gamma_3}, \Gamma \vdash \gamma_4 \preceq \gamma_5$*.*
4. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+,\gamma_2,\gamma_3} \Gamma \vdash \gamma_4 \subseteq \gamma_5$ *and either* $\gamma_5 \in \mathbf{Dom}_C(\Gamma)$ *or* $\Psi \vdash \gamma_5 \subseteq \gamma_3$*, then* $\Psi \mid^{\overrightarrow{\gamma_1}^+,\gamma_2} \blacktriangleright^{\gamma_3}, \Gamma \vdash \gamma_4 \subseteq \gamma_5$*.*
5. *If* $\Psi \mid^{\overrightarrow{\gamma_1}^+,\gamma_2,\gamma_3} \Gamma \vdash_{\mathbf{id}}^{\overrightarrow{\gamma_4}^+} e : A$*, then* $\Psi \mid^{\overrightarrow{\gamma_1}^+,\gamma_2} \blacktriangleright^{\gamma_3}, \Gamma \vdash_{\mathbf{id}}^{\overrightarrow{\gamma_4}^+} e : A$*.*

Proof. By mutual induction on the given derivation. □

Lemma 35 (Demotion for $\blacktriangleleft$)**.** *Suppose* $\gamma_2(:\succeq \delta_0/ :\supseteq\ \delta_k) \in \Psi$*,* $\Psi \vdash \overrightarrow{\gamma_1} \# \gamma_2$*, and* $\gamma_4 \notin \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma)$*.*

1. *If* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash^{\overrightarrow{\gamma_3}^+} \mathbf{wf}$*, then* $\Psi \mid^{\overrightarrow{\gamma_1},\delta_0,\ldots,\delta_k} \blacktriangleleft_k^{\gamma_4}, \Gamma[\gamma_2 := \gamma_4] \vdash^{\overrightarrow{\gamma_3}^+[\gamma_2:=\gamma_4]} \mathbf{wf}$*.*
2. *If* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash^{\overrightarrow{\gamma_3}^+} A\ \mathbf{wf}$*, then* $\Psi \mid^{\overrightarrow{\gamma_1},\delta_0,\ldots,\delta_k} \blacktriangleleft_k^{\gamma_4}, \Gamma[\gamma_2 := \gamma_4] \vdash^{\overrightarrow{\gamma_3}^+[\gamma_2:=\gamma_4]} A[\gamma_2 := \gamma_4]\ \mathbf{wf}$*.*
3. *If* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash \gamma_5 \preceq \gamma_6$ *and either* $\gamma_6 \in \mathbf{Dom}_C(\Gamma)$ *or there exists* $\gamma_7 \in \{\overrightarrow{\gamma_1}, \gamma_2\}$ *such that* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash \gamma_6 \subseteq \gamma_7$*, then* $\Psi \mid^{\overrightarrow{\gamma_1},\delta_0,\ldots,\delta_k} \blacktriangleleft_k^{\gamma_4}, \Gamma[\gamma_2 := \gamma_4] \vdash \gamma_5[\gamma_2 := \gamma_4] \preceq \gamma_6[\gamma_2 := \gamma_4]$*.*
4. *If* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash \gamma_5 \subseteq \gamma_6$ *and either* $\gamma_6 \in \mathbf{Dom}_C(\Gamma)$ *or there exists* $\gamma_7 \in \{\overrightarrow{\gamma_1}, \gamma_2\}$ *such that* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash \gamma_6 \subseteq \gamma_7$*, then* $\Psi \mid^{\overrightarrow{\gamma_1},\delta_0,\ldots,\delta_k} \blacktriangleleft_k^{\gamma_4}, \Gamma[\gamma_2 := \gamma_4] \vdash \gamma_5[\gamma_2 := \gamma_4] \subseteq \gamma_6[\gamma_2 := \gamma_4]$*.*
5. *If* $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash_{\mathbf{id}}^{\overrightarrow{\gamma_3}^+} e : A$*, then* $\Psi \mid^{\overrightarrow{\gamma_1},\delta_0,\ldots,\delta_k} \blacktriangleleft_k^{\gamma_4}, \Gamma[\gamma_2 := \gamma_4] \vdash_{\mathbf{id}}^{\overrightarrow{\gamma_3}^+[\gamma_2:=\gamma_4]} e : A[\gamma_2 := \gamma_4]$*.*

Proof. We proceed by mutual induction on the given derivation. The proof follows the same pattern as Lemma 33; we record the points where the $\blacktriangleleft$ entry is used.

Case Item 1**:** By induction on the context well-formedness derivation. The case for R-WF-Ctx-$\blacktriangleright$ is the only one that uses a visibility premise. By inversion, the last rule gives premises

$$\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1 \vdash^{\overrightarrow{\delta_1'},\delta_2'} \mathbf{wf} \quad \text{and} \quad \Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma_1 \vdash \delta_3' \subseteq \delta_2'.$$

The induction hypothesis gives the demoted well-formedness of Γ_1. By Lemma 22, $\delta_2' \in \{\overrightarrow{\gamma_1},\gamma_2\} \cup \mathbf{Dom}_C(\Gamma_1) \cup \mathbf{FC}(\Gamma_1)$, so Item 4 gives the demoted visibility premise. Reapplying R-WF-Ctx-$\blacktriangleright$ completes the case.

Case Item 2**:** By induction on the type well-formedness derivation.

Case Item 3**:** This case is analogous to Item 4, using the rules and inversion principles for $\preceq$ in place of those for $\subseteq$.

Case Item 4**:** By induction with respect to $\mathbf{deg}(\Psi,\Gamma,\gamma_6)$. By the side condition of the lemma, either $\gamma_6 \in \mathbf{Dom}_C(\Gamma)$ or there exists $\gamma_7 \in \{\overrightarrow{\gamma_1},\gamma_2\}$ such that $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash \gamma_6 \subseteq \gamma_7$.

We first handle the case $\gamma_6 \notin \mathbf{Dom}_C(\Gamma)$. Then there exists $\gamma_7 \in \{\overrightarrow{\gamma_1},\gamma_2\}$ such that $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash \gamma_6 \subseteq \gamma_7$. We have $\gamma_6 \in \mathbf{Dom}_C(\Psi)$ by regularity; hence $\Psi \vdash \gamma_5 \subseteq \gamma_6$ by Lemma 5. Moreover, regularity and Item 1 give

$$\Psi \mid^{\overrightarrow{\gamma_1},\delta_0,\ldots,\delta_k} \blacktriangleleft_k^{\gamma_4}, \Gamma[\gamma_2 \coloneqq \gamma_4] \vdash^{\cdots} \mathbf{wf}.$$

Therefore, by Lemma 6,

$$\Psi \mid^{\overrightarrow{\gamma_1},\delta_0,\ldots,\delta_k} \blacktriangleleft_k^{\gamma_4}, \Gamma[\gamma_2 \coloneqq \gamma_4] \vdash \gamma_5 \subseteq \gamma_6.$$

We conclude by cases on whether the substitution $\gamma_2 \coloneqq \gamma_4$ affects γ_5 or γ_6.

Case $\gamma_2 \neq \gamma_5$ and $\gamma_2 \neq \gamma_6$**:** The substitution leaves both endpoints unchanged, so the required judgment follows immediately.

Case $\gamma_2 \neq \gamma_5$ and $\gamma_2 = \gamma_6$**:** By Lemma 9 and $\gamma_2(:\succeq \delta_0/ :\supseteq \ \delta_k) \in \Psi$, we have $\Psi \vdash \gamma_5 \subseteq \delta_0$ or $\Psi \vdash \gamma_5 \subseteq \delta_k$. By Lemma 6, the corresponding judgment holds in the demoted context. If the intermediate classifier is δ_0, then R-$\preceq$-$\blacktriangleleft$-Ctx followed by R-$\subseteq$-Lift gives

$$\Psi \mid^{\overrightarrow{\gamma_1},\delta_0,\ldots,\delta_k} \blacktriangleleft_k^{\gamma_4}, \Gamma[\gamma_2 \coloneqq \gamma_4] \vdash \delta_0 \subseteq \gamma_4.$$

If the intermediate classifier is δ_k, then R-$\subseteq$-$\blacktriangleleft$-Ctx gives

$$\Psi \mid^{\overrightarrow{\gamma_1},\delta_0,\ldots,\delta_k} \blacktriangleleft_k^{\gamma_4}, \Gamma[\gamma_2 \coloneqq \gamma_4] \vdash \delta_k \subseteq \gamma_4.$$

In either subcase, the conclusion follows by R-$\subseteq$-Trans.

Case $\gamma_2 = \gamma_5 = \gamma_6$**:** Then $\gamma_5[\gamma_2 \coloneqq \gamma_4] = \gamma_6[\gamma_2 \coloneqq \gamma_4] = \gamma_4$. The required judgment follows from R-$\preceq$-Refl and R-$\subseteq$-Lift.

Case $\gamma_2 = \gamma_5$ and $\gamma_2 \neq \gamma_6$**:** Since $\gamma_5 = \gamma_2$, we have $\Psi \vdash \gamma_2 \subseteq \gamma_6$. Moreover, for some $\gamma_7 \in \{\overrightarrow{\gamma_1},\gamma_2\}$ we have $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash \gamma_6 \subseteq \gamma_7$. By Lemma 5, this gives $\Psi \vdash \gamma_6 \subseteq \gamma_7$; together with $\Psi \vdash \gamma_2 \subseteq \gamma_6$ and R-$\subseteq$-Trans, we get $\Psi \vdash \gamma_2 \subseteq \gamma_7$. Since $\Psi \vdash \overrightarrow{\gamma_1} \ \# \ \gamma_2$, the only possible such γ_7 is γ_2. Hence $\Psi \vdash \gamma_6 \subseteq \gamma_2$. Applying Lemma 4 to $\Psi \vdash \gamma_2 \subseteq \gamma_6$ and $\Psi \vdash \gamma_6 \subseteq \gamma_2$ yields $\gamma_2 = \gamma_6$, contradicting the case assumption. Thus this case is impossible.

It remains to consider the case $\gamma_6 \in \mathbf{Dom}_C(\Gamma)$. If $\gamma_5 = \gamma_6$, then $\gamma_6[\gamma_2 \coloneqq \gamma_4]$ is declared in the demoted context by Item 1. Hence the desired judgment follows from R-$\preceq$-Refl and R-$\subseteq$-Lift. Thus assume $\gamma_5 \neq \gamma_6$. We perform case analysis using Lemma 10. In each case, the inversion gives a derivation $\Psi \mid^{\overrightarrow{\gamma_1},\gamma_2} \Gamma \vdash \gamma_5 \subseteq \delta_3'$ where $\mathbf{deg}(\Psi,\Gamma,\delta_3') < \mathbf{deg}(\Psi,\Gamma,\gamma_6)$ and $\delta_3' \in \{\overrightarrow{\gamma_1},\gamma_2\} \cup \mathbf{Dom}_C(\Gamma) \cup \mathbf{FC}(\Gamma)$. The membership condition follows from Lemma 22 when δ_3' comes from the output stack of the corresponding context prefix. In the remaining polymorphic-classifier declaration cases, δ_3' appears as the right-hand side of a declaration $\gamma_6 :\succeq \delta_3'$, so the condition follows from the definitions of $\mathbf{Dom}_C(\Gamma)$ and $\mathbf{FC}(\Gamma)$. Hence, the induction hypothesis gives

$$\Psi \mid^{\overrightarrow{\gamma_1},\delta_0,\ldots,\delta_k} \blacktriangleleft_k^{\gamma_4}, \Gamma[\gamma_2 \coloneqq \gamma_4] \vdash \gamma_5[\gamma_2 \coloneqq \gamma_4] \subseteq \delta_3'[\gamma_2 \coloneqq \gamma_4].$$

Reapplying the corresponding context rule gives

$$\Psi \mid^{\overrightarrow{\gamma_1},\delta_0,\ldots,\delta_k} \blacktriangleleft_k^{\gamma_4}, \Gamma[\gamma_2 \coloneqq \gamma_4] \vdash \delta_3'[\gamma_2 \coloneqq \gamma_4] \subseteq \gamma_6[\gamma_2 \coloneqq \gamma_4].$$

The desired judgment follows by R-$\subseteq$-Trans.

Case Item 5**:** By induction on the typing derivation. The cases R-T-Var, R-T-Name, and R-T-Splice are analogous to the corresponding cases in Lemma 33, using Item 1 for context well-formedness and Item 3 for reachability premises.

□

Lemma 36 (Demotion for Polymorphic Classifier Declarations)**.** *Let* $\sigma = \gamma_0 \coloneqq \delta'_0, \ldots, \gamma_k \coloneqq \delta'_k$*. Suppose* $[\gamma_0 :\succeq \gamma'_0, \ldots, \gamma_k :\succeq \gamma'_k] \in \Psi$*,* $\Psi \vdash \overrightarrow{\delta_1}^+ \# \gamma_0, \ldots, \gamma_k$*, and* $\delta'_i \notin \mathbf{Dom}_C(\Psi) \cup \mathbf{Dom}_C(\Gamma)$ *for* $0 \leq i \leq k$*.*

1. *If* $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma_0}\, \Gamma \vdash^{\overrightarrow{\delta_2}^+} \mathbf{wf}$*, then* $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma'_0}\, [\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_k :\succeq \gamma'_k]^{\delta'_0}, \Gamma[\sigma] \vdash^{\overrightarrow{\delta_2}^+[\sigma]} \mathbf{wf}$*.*
2. *If* $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma_0}\, \Gamma \vdash^{\overrightarrow{\delta_2}^+} A\ \mathbf{wf}$*, then* $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma'_0}\, [\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_k :\succeq \gamma'_k]^{\delta'_0}, \Gamma[\sigma] \vdash^{\overrightarrow{\delta_2}^+[\sigma]} A[\sigma]\ \mathbf{wf}$*.*
3. *If* $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma_0}\, \Gamma \vdash \delta_2 \preceq \delta_3$ *and either* $\delta_3 \in \mathbf{Dom}_C(\Gamma)$ *or there exists* $\delta_4 \in \{\overrightarrow{\delta_1}^+, \gamma_0\}$ *such that* $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma_0}\, \Gamma \vdash \delta_3 \preceq \delta_4$*, then* $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma'_0}\, [\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_k :\succeq \gamma'_k]^{\delta'_0}, \Gamma[\sigma] \vdash \delta_2[\sigma] \preceq \delta_3[\sigma]$*.*
4. *If* $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma_0}\, \Gamma \vdash \delta_2 \subseteq \delta_3$ *and either* $\delta_3 \in \mathbf{Dom}_C(\Gamma)$ *or there exists* $\delta_4 \in \{\overrightarrow{\delta_1}^+, \gamma_0\}$ *such that* $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma_0}\, \Gamma \vdash \delta_3 \subseteq \delta_4$*, then* $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma'_0}\, [\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_k :\succeq \gamma'_k]^{\delta'_0}, \Gamma[\sigma] \vdash \delta_2[\sigma] \subseteq \delta_3[\sigma]$*.*
5. *If* $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma_0}\, \Gamma \vdash^{\overrightarrow{\delta_2}^+}_{\mathbf{id}} e : A$*, then* $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma'_0}\, [\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_k :\succeq \gamma'_k]^{\delta'_0}, \Gamma[\sigma] \vdash^{\overrightarrow{\delta_2}^+[\sigma]}_{\mathbf{id}} e : A[\sigma]$*.*

Proof. We proceed by mutual induction on the given derivation. The proof follows the same pattern as Lemma 33; the main difference is that the demotion substitutes the whole sequence of classifiers in σ and introduces the polymorphic-classifier declaration $[\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_k :\succeq \gamma'_k]^{\delta'_0}$ in the local context.

Case Item 1**:** This case is analogous to Item 1 of Lemma 33.

Case Item 2**:** By induction on the type well-formedness derivation.

Case Item 3**:** This case is analogous to Item 4, using the rules and inversion principles for $\preceq$ in place of those for $\subseteq$.

Case Item 4**:** By induction with respect to $\mathbf{deg}(\Psi, \Gamma, \delta_3)$. By the side condition of the lemma, either $\delta_3 \in \mathbf{Dom}_C(\Gamma)$ or there exists $\delta_4 \in \{\overrightarrow{\delta_1}^+, \gamma_0\}$ such that $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma_0}\, \Gamma \vdash \delta_3 \subseteq \delta_4$.

We first handle the case $\delta_3 \notin \mathbf{Dom}_C(\Gamma)$. Then there exists $\delta_4 \in \{\overrightarrow{\delta_1}^+, \gamma_0\}$ such that $\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma_0}\, \Gamma \vdash \delta_3 \subseteq \delta_4$. From disjointness of $\mathbf{Dom}_C(\Gamma)$ and $\mathbf{Dom}_C(\Psi)$, we have $\delta_3 \in \mathbf{Dom}_C(\Psi)$. Hence, by Lemma 5, we have $\Psi \vdash \delta_2 \subseteq \delta_3$. Moreover, regularity and Item 1 give

$$\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma'_0}\, [\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_k :\succeq \gamma'_k]^{\delta'_0}, \Gamma[\sigma] \vdash^{\cdots} \mathbf{wf}.$$

Therefore, by Lemma 6, we obtain

$$\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma'_0}\, [\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_k :\succeq \gamma'_k]^{\delta'_0}, \Gamma[\sigma] \vdash \delta_2 \subseteq \delta_3.$$

We conclude by cases on whether σ affects δ_2 or δ_3.

Case $\delta_2 \notin \{\gamma_0, \ldots, \gamma_k\}$ and $\delta_3 \notin \{\gamma_0, \ldots, \gamma_k\}$**:** The substitution leaves both endpoints unchanged, so the required judgment follows immediately.

Case $\delta_2 \notin \{\gamma_0, \ldots, \gamma_k\}$ and $\delta_3 \in \{\gamma_0, \ldots, \gamma_k\}$**:** Let $\delta_3 = \gamma_i$. If $\delta_2 = \gamma_i$, this contradicts the assumption $\delta_2 \notin \{\gamma_0, \ldots, \gamma_k\}$. Thus $\delta_2 \neq \gamma_i$. By Lemma 9, the derivation $\Psi \vdash \delta_2 \subseteq \gamma_i$ factors through either γ'_i or, when $i < k$, γ_{i+1}. In the former subcase, Lemma 6 gives the demoted-context derivation ending in γ'_i, and the polymorphic-classifier declaration introduced in the context gives $\gamma'_i \subseteq \delta'_i$. In the latter subcase, the induction hypothesis applies to the smaller derivation ending in γ_{i+1}; the same context declaration gives $\delta'_{i+1} \subseteq \delta'_i$. Hence in either subcase we obtain

$$\Psi \,|^{\overrightarrow{\delta_1}^+, \gamma'_0}\, [\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_k :\succeq \gamma'_k]^{\delta'_0}, \Gamma[\sigma] \vdash \delta_2[\sigma] \subseteq \delta'_i,$$

which is the required judgment because $\gamma_i[\sigma] = \delta'_i$.

Case $\delta_2 \in \{\gamma_0, \ldots, \gamma_k\}$ and $\delta_3 \in \{\gamma_0, \ldots, \gamma_k\}$**:** Let $\delta_2 = \gamma_i$ and $\delta_3 = \gamma_j$. Since $\Psi \vdash \gamma_i \subseteq \gamma_j$, the order of the polymorphic classifier declaration gives $i \geq j$. If $i = j$, the result follows by reflexivity and lifting after substitution. If $i > j$, reapplying the corresponding polymorphic-classifier context rule along the sequence from δ'_i to δ'_j gives

$$\Psi \mid^{\overrightarrow{\delta_1}^+, \gamma'_0} [\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_k :\succeq \gamma'_k]^{\delta'_0}, \Gamma[\sigma] \vdash \delta'_i \subseteq \delta'_j.$$

This is exactly the required judgment.

Case $\delta_2 \in \{\gamma_0, \ldots, \gamma_k\}$ and $\delta_3 \notin \{\gamma_0, \ldots, \gamma_k\}$**:** Let $\delta_2 = \gamma_i$. Then $\Psi \vdash \gamma_i \subseteq \delta_3$. Moreover, for some $\delta_4 \in \{\overrightarrow{\delta_1}^+, \gamma_0\}$ we have $\Psi \mid^{\overrightarrow{\delta_1}^+, \gamma_0} \Gamma \vdash \delta_3 \subseteq \delta_4$. By Lemma 5, this gives $\Psi \vdash \delta_3 \subseteq \delta_4$; together with $\Psi \vdash \gamma_i \subseteq \delta_3$ and R-$\subseteq$-Trans, we get $\Psi \vdash \gamma_i \subseteq \delta_4$. If $\delta_4 \in \{\overrightarrow{\delta_1}^+\}$, this contradicts $\Psi \vdash \overrightarrow{\delta_1}^+ \# \gamma_0, \ldots, \gamma_k$. Hence $\delta_4 = \gamma_0$, and so $\Psi \vdash \delta_3 \subseteq \gamma_0$. Now $\Psi \vdash \gamma_i \subseteq \delta_3$ and $\Psi \vdash \delta_3 \subseteq \gamma_0$ place δ_3 between γ_i and γ_0 in the same polymorphic-classifier declaration. By Lemma 3, this forces $\delta_3 = \gamma_j$ for some j, contradicting $\delta_3 \notin \{\gamma_0, \ldots, \gamma_k\}$. Thus this case is impossible.

It remains to consider the case $\delta_3 \in \mathbf{Dom}_C(\Gamma)$. If $\delta_2 = \delta_3$, then $\delta_3[\sigma]$ is declared in the demoted context by Item 1. Hence the desired judgment follows from R-$\preceq$-Refl and R-$\subseteq$-Lift. Thus assume $\delta_2 \neq \delta_3$. We perform case analysis using Lemma 10. In each case, the inversion gives a derivation $\Psi \mid^{\overrightarrow{\delta_1}^+, \gamma_0} \Gamma \vdash \delta_2 \subseteq \delta_4$ where $\mathbf{deg}(\Psi, \Gamma, \delta_4) < \mathbf{deg}(\Psi, \Gamma, \delta_3)$ and $\delta_4 \in \{\overrightarrow{\delta_1}^+, \gamma_0\} \cup \mathbf{Dom}_C(\Gamma) \cup \mathbf{FC}(\Gamma)$. The membership condition follows from Lemma 22 when δ_4 comes from the output stack of the corresponding context prefix. In the remaining polymorphic-classifier declaration cases, δ_4 appears as the right-hand side of a declaration $\delta_3 :\succeq \delta_4$, so the condition follows from the definitions of $\mathbf{Dom}_C(\Gamma)$ and $\mathbf{FC}(\Gamma)$. Hence, the induction hypothesis gives

$$\Psi \mid^{\overrightarrow{\delta_1}^+, \gamma'_0} [\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_k :\succeq \gamma'_k]^{\delta'_0}, \Gamma[\sigma] \vdash \delta_2[\sigma] \subseteq \delta_4[\sigma].$$

Reapplying the corresponding context rule gives

$$\Psi \mid^{\overrightarrow{\delta_1}^+, \gamma'_0} [\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_k :\succeq \gamma'_k]^{\delta'_0}, \Gamma[\sigma] \vdash \delta_4[\sigma] \subseteq \delta_3[\sigma].$$

The desired judgment follows by R-$\subseteq$-Trans.

Case Item 5**:** By induction on the typing derivation. The cases R-T-Var, R-T-Name, and R-T-Splice are analogous to the corresponding cases in Lemma 33, using Item 1 for context well-formedness and Item 3 for reachability premises. The cases for R-T-Polycls and R-T-Appcls additionally use Lemma 16 to commute the classifier substitution through polymorphic classifier declarations.

$\square$

Corollary 4 (Demotion)**.** *1. Suppose* $\Psi \mid^{\overrightarrow{\gamma_1}^+, \gamma_2, \gamma_3} \varepsilon \vdash_{\mathbf{id}}^{\overrightarrow{\gamma_1}^+, \gamma_2, \gamma_3} e : A$*. If* $\Psi \vdash \gamma_3 \subseteq \gamma_2$*, then* $\Psi \mid^{\overrightarrow{\gamma_1}^+, \gamma_2} \blacktriangleright^{\gamma_3} \vdash_{\mathbf{id}}^{\overrightarrow{\gamma_1}^+, \gamma_2, \gamma_3} e : A$*.*

2. *Suppose* $\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} \varepsilon \vdash_{\mathbf{id}}^{\overrightarrow{\gamma_1}, \gamma_2} e : A$ *and* $\gamma_2(:\succeq \delta_0 / :\supseteq \delta_k) \in \Psi$ *and* $\Psi \vdash \overrightarrow{\gamma_1} \# \gamma_2$*. If* $\gamma_4 \notin \mathbf{Dom}_C(\Psi)$*, then* $\Psi \mid^{\overrightarrow{\gamma_1}, \delta_0, \ldots, \delta_k} \blacktriangleleft_k^{\gamma_4} \vdash_{\mathbf{id}}^{\overrightarrow{\gamma_1}, \gamma_4} e : A[\gamma_2 := \gamma_4]$*.*

3. *Suppose* $\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_2} \varepsilon \vdash_{\mathbf{id}}^{\overrightarrow{\gamma_1}, \gamma_2} e : A_2$ *and* $x : A_1 \mathbin{@} \gamma_2 :\succeq \gamma_4 \in \Psi$ *and* $\Psi \vdash \overrightarrow{\gamma_1} \# \gamma_2$*. If* $\gamma_5 \notin \mathbf{Dom}_C(\Psi)$*, then* $\Psi \mid^{\overrightarrow{\gamma_1}, \gamma_4} x :^{\gamma_5} A_1[\gamma_2 := \gamma_5] \vdash_{\mathbf{id}}^{\overrightarrow{\gamma_1}, \gamma_5} e : A_2[\gamma_2 := \gamma_5]$*.*

4. *Let* $\sigma = \gamma_0 := \delta'_0, \ldots, \gamma_k := \delta'_k$*. Suppose* $\Psi \mid^{\overrightarrow{\delta_1}^+, \gamma_0} \varepsilon \vdash_{\mathbf{id}}^{\overrightarrow{\delta_1}^+, \gamma_0} e : A$*, and* $[\gamma_0 :\succeq \gamma'_0, \ldots, \gamma_k :\succeq \gamma'_k] \in \Psi$ *and* $\Psi \vdash \overrightarrow{\delta_1}^+ \# \gamma_0, \ldots, \gamma_k$*. If* $\delta'_i \notin \mathbf{Dom}_C(\Psi)$ *for* $0 \leq i \leq k$*, then* $\Psi \mid^{\overrightarrow{\delta_1}^+, \gamma'_0} [\delta'_1 :\succeq \gamma'_1, \ldots, \delta'_k :\succeq \gamma'_k]^{\delta'_0} \vdash_{\mathbf{id}}^{\overrightarrow{\delta_1}^+, \delta'_0} e : A[\sigma]$*.*

3.9 Type Soundness

Lemma 37 (Inversion for Runtime Expression Typing)**.** *Suppose* $\Psi \mid^{\overrightarrow{\gamma_1}^+} \Gamma \vdash_R^{\overrightarrow{\gamma_2}^+} e : A$*.*

1. *If* $e = n$*, then the derivation ends with R-T-Int.*

2. *If* $e = x$*, then the derivation ends with R-T-Appcls, R-T-Var or R-T-Name.*

3. *If* $e = \lambda x.\ e'$ *and* $A = A_1 \to A_2$*, then the derivation ends with R-T-Func.*

4. *If* $e = \lambda x.\ e'$ *and* $A = \forall \varrho.A'$*, then the derivation ends with R-T-Polycls.*

5. *If* $e = \mathbf{rec}\ f(x).\ e'$ *and* $A = A_1 \to A_2$*, then the derivation ends with R-T-Rec.*

6. *If* $e = \mathbf{rec}\ f(x).\ e'$ *and* $A = \forall \varrho.A'$*, then the derivation ends with R-T-Polycls.*

7. *If* $e = \langle e' \rangle$ *and* $A = \langle A' \rangle^{\gamma_3}$*, then the derivation ends with R-T-Quote.*

8. *If* $e = \langle e' \rangle$ *and* $A = \forall \varrho.A'$*, then the derivation ends with R-T-Polycls.*

9. *If* $e = e_1\, e_2$*, then the derivation ends with R-T-Appcls or R-T-App.*

10. *If* $e = \$_k\{e'\}$*, then the derivation ends with R-T-Appcls or R-T-Splice.*

11. *If* $e = \mathbf{ref}\,(e')$*, then the derivation ends with R-T-Ref.*

12. *If* $e = \mathbf{deref}(e')$*, then the derivation ends with R-T-Appcls or R-T-Deref.*

13. *If* $e = e_1 \coloneqq e_2$*, then the derivation ends with R-T-Appcls or R-T-Assign.*

Proof. This follows directly from Lemma 19. □

Lemma 38 (Safety of **lookup**)**.** *If* $\Psi \mid^{\gamma} \varepsilon \vdash_R^{\gamma} x : A$*, which is derived by R-T-Name, and* $\Psi \mid \Theta \vdash^{\gamma} E\ \mathbf{wf}$*, then there exists* v *such that* $\mathbf{lookup}\,(E, \mathbf{rename}\,(R, x)) = \mathbf{Done}\,(v)$ *and* $\Psi \mid \Theta \vdash^{\gamma} v : A\ \mathbf{val}$*.*

Proof. By induction on the derivation of $\Psi \mid \Theta \vdash^{\gamma} E\ \mathbf{wf}$. For the base case where $E = (E_1, x \coloneqq v)$, we use Lemma 27 to conclude. □

Lemma 39 (Safety of **lookupStore**)**.** *If* $\Psi \mid \Theta \vdash^{\gamma} l : A\ \mathbf{ref}\ \mathbf{val}$ *and* $\Psi \vdash S \colon \Theta$*, then there exists* v *such that* $\mathbf{lookupStore}(S, l) = \mathbf{Done}\,(v)$ *and* $\Psi \mid \Theta \vdash^{\gamma} v : A\ \mathbf{val}$*.*

Proof. Straightforward by inverting VT-Loc and T-Store. We also use Lemma 27 to conclude. □

Lemma 40 (Safety of **updateStore**)**.** *If* $\Psi \mid \Theta \vdash^{\gamma} v : A\ \mathbf{val}$*,* $\Psi \vdash S \colon \Theta$ *and* $l : A \in \Theta$*, then* $\mathbf{updateStore}\,(S, l, v) = \mathbf{Done}\,(S_1)$ *for some* S_1 *where* $\Psi \vdash S_1 \colon \Theta$*.*

Proof. Straightforward by inverting T-Store. □

Theorem 1 (Soundness of Runtime Typing)**.**

1. *If* $\Psi_1 \mid^{\gamma_1} \varepsilon \vdash_R^{\gamma_1} e^0 : A$*,* $\Psi_1 \mid \Theta_1 \vdash^{\gamma_1} E\ \mathbf{wf}$*, and* $\Psi_1 \vdash S_1 \colon \Theta_1$*, then either of the following holds.*
   - $\mathcal{V}^{\mathrm{rt}}(e^0, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2) = \mathbf{Timeout}$
   - *There exist* v*,* Ψ_2*,* Θ_2*, and* S_2 *such that*
     - $\mathcal{V}^{\mathrm{rt}}(e^0, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2) = \mathbf{Done}\,(v, \mathbf{Dom}_V(\Psi_2), S_2)$*,*
     - $\Psi_2 \mid \Theta_2 \vdash^{\gamma_1} v : A\ \mathbf{val}$*,*
     - $\Psi_2 \vdash S_2 \colon \Theta_2$*, and*
     - $\Psi_1 \subseteq \Psi_2$ *and* $\Theta_1 \subseteq \Theta_2$*.*

2. *If* $\Psi_1 \mid^{\gamma_0,\ldots,\gamma_{k_1+}} \varepsilon \vdash_R^{\gamma_0,\ldots,\gamma_{k_1+}} {e_1}^{k_1+} : A$*,* $\Psi_1 \mid \Theta_1 \vdash^{\gamma_0} E\ \mathbf{wf}$*, and* $\Psi_1 \vdash S_1 \colon \Theta_1$*, then either of the following holds.*
   - $\mathcal{V}^{\mathrm{fut}}(k_1, {e_1}^{k_1+}, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2) = \mathbf{Timeout}$
   - *There exist* ${e_2}^{k_1}$*,* Ψ_2*,* Θ_2*, and* S_2 *such that*
     - $\mathcal{V}^{\mathrm{fut}}(k_1, {e_1}^{k_1+}, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2) = \mathbf{Done}\,({e_2}^{k_1}, \mathbf{Dom}_V(\Psi_2), S_2)$*,*
     - $\Psi_2 \mid^{\gamma_1,\ldots,\gamma_{k_1+}} \varepsilon \vdash_{\mathbf{id}}^{\gamma_1,\ldots,\gamma_{k_1+}} {e_2}^{k_1} : A$*,*
     - $\Psi_2 \vdash S_2 \colon \Theta_2$*, and*
     - $\Psi_1 \subseteq \Psi_2$ *and* $\Theta_1 \subseteq \Theta_2$*.*

Proof. By mutual induction with respect to k_2 and the given derivation.

Item 1: The case where $k_2 = 0$ is trivial. We consider the case where $k_2 > 0$. We perform case analysis on e.

Case Last derivation is R-T-Appcls**:** Let $\varrho = \delta_1 :\succeq \delta'_1, \ldots, \delta_k :\succeq \delta'_k$ and $\sigma = \delta_1 := \delta''_1, \ldots, \delta_k := \delta''_k$. We are given

$$\textsc{R-T-Appcls}\ \frac{\Psi_1 \mid^{\gamma_1} \varepsilon \vdash^{\gamma_1}_R e^0 : \forall \varrho. A' \qquad \Psi_1 \mid^{\gamma_1} \varepsilon \vdash \delta'_i \preceq \delta''_i \text{ for } 1 \leq i \leq k \qquad \Psi_1 \mid^{\gamma_1} \varepsilon \vdash \delta''_{i+1} \subseteq \delta''_i \text{ for } 0 \leq i < k}{\Psi_1 \mid^{\gamma_1} \varepsilon \vdash^{\gamma_1}_R e^0 : A'[\sigma]}$$

where $\delta''_0 = \gamma_1$ and $A = A'[\sigma]$.

By the induction hypothesis applied to the premise typing derivation, either of the following holds:

- $\mathcal{V}^{\mathrm{rt}}(e^0, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2) = \mathbf{Timeout}$
- We have v_1, Ψ_2, S_2 and Θ_2 such that
  - $\mathcal{V}^{\mathrm{rt}}(e^0, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2) = \mathbf{Done}\,(v_1, \mathbf{Dom}_V(\Psi_2), S_2)$
  - $\Psi_2 \mid \Theta_2 \vdash^{\gamma_1} v_1 : \forall \varrho. A'$ **val**
  - $\Psi_2 \vdash S_2 \colon \Theta_2$
  - $\Psi_1 \subseteq \Psi_2$
  - $\Theta_1 \subseteq \Theta_2$

The first case is trivial, and we consider the second case. By weakening along $\Psi_1 \subseteq \Psi_2$, we have

- $\Psi_2 \mid^{\gamma_1} \varepsilon \vdash \delta'_i \preceq \delta''_i$ for each i with $1 \leq i \leq k$,
- $\Psi_2 \mid^{\gamma_1} \varepsilon \vdash \delta''_{i+1} \subseteq \delta''_i$ for each i with $0 \leq i < k$.

By Lemma 5, these give the global premises required by Lemma 28. Hence, by Lemma 28, there exists Ψ_3 such that $\Psi_2 \subseteq \Psi_3$, $\mathbf{Dom}_V(\Psi_3) = \mathbf{Dom}_V(\Psi_2)$, and

$$\Psi_3 \mid \Theta_2 \vdash^{\gamma_1} v_1 : A'[\sigma] \ \mathbf{val}.$$

Moreover, $\Psi_3 \vdash S_2 \colon \Theta_2$ follows by weakening along $\Psi_2 \subseteq \Psi_3$. Thus the desired result follows by taking the final value to be v_1, the final reserved-name context to be Ψ_3, and the final store typing context to be Θ_2.

Case $e^0 = x$**:** By Lemma 37, the typing derivation ends with R-T-Var, R-T-Name, or R-T-Appcls.

The case of R-T-Var is impossible because $\Gamma = \varepsilon$. The case of R-T-Appcls has already been discussed. The remaining case, R-T-Name, follows from Lemma 38.

Case $e^0 = \lambda x.\ e^0$**:** By VT-Clos-Func.

Case $e^0 = \mathbf{rec}\ f(x).\ e^0$**:** By VT-Clos-Rec.

Case $e^0 = {e_1}^0 {e_2}^0$**:** By Lemma 37, it is derived by R-T-Appcls or R-T-App. As the former case is already discussed, we consider the latter. In this case, we are given

$$\textsc{R-T-App}\ \frac{\Psi_1 \mid^{\gamma_1} \varepsilon \vdash^{\gamma_1}_R {e_1}^0 : B \to A \qquad \Psi_1 \mid^{\gamma_1} \varepsilon \vdash^{\gamma_1}_R {e_2}^0 : B}{\Psi_1 \mid^{\gamma_1} \varepsilon \vdash^{\gamma_1}_R {e_1}^0 {e_2}^0 : A}$$

for some B. By IH on ${e_1}^0$, either of the following holds:

- $\mathcal{V}^{\mathrm{rt}}({e_1}^0, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2 - 1) = \mathbf{Timeout}$
- We have v_1, Ψ_2, S_2 and Θ_2 such that
  - $\mathcal{V}^{\mathrm{rt}}({e_1}^0, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2 - 1) = \mathbf{Done}\,(v_1, \mathbf{Dom}_V(\Psi_2), S_2)$
  - $\Psi_2 \mid \Theta_2 \vdash^{\gamma_1} v_1 : B \to A$ **val**
  - $\Psi_2 \vdash S_2 \colon \Theta_2$
  - $\Psi_1 \subseteq \Psi_2$
  - $\Theta_1 \subseteq \Theta_2$

The first case is trivial, and we consider the second case. We have $\Psi_2 \mid^{\gamma_1} \varepsilon \vdash^{\gamma_1}_{R} {e_2}^0 : B$ by Lemma 11. Moreover, $\Psi_2 \mid \Theta_2 \vdash^{\gamma_1} E$ **wf** follows by weakening along $\Psi_1 \subseteq \Psi_2$ and $\Theta_1 \subseteq \Theta_2$. Thus, we can apply the IH and get either of the following.

- $\mathcal{V}^{\mathrm{rt}}({e_2}^0, E, R, \mathbf{Dom}_V(\Psi_2), S_2, k_2 - 1) = \mathbf{Timeout}$
- We have v_2, Ψ_3, S_3 and Θ_3 such that
  - $\mathcal{V}^{\mathrm{rt}}({e_2}^0, E, R, \mathbf{Dom}_V(\Psi_2), S_2, k_2 - 1) = \mathbf{Done}\,(v_2, \mathbf{Dom}_V(\Psi_3), S_3)$
  - $\Psi_3 \mid \Theta_3 \vdash^{\gamma_1} v_2 : B$ **val**
  - $\Psi_3 \vdash S_3 \colon \Theta_3$
  - $\Psi_2 \subseteq \Psi_3$
  - $\Theta_2 \subseteq \Theta_3$

The first case is trivial, and we consider the second case. v_1 is typed by either of VT-Clos-Func or VT-Clos-Rec. We consider the case where v_1 is typed by VT-Clos-Rec, and the other case is similar. By inverting the typing of v_1, we have $v_1 = \mathbf{clos}\,(E_1, R_1, \mathbf{rec}\ f(x).\ {e_3}^0)$ for some E_1, R_1, f, x and ${e_3}^0$ such that

- $\Psi_3 \mid^{\gamma_2} \varepsilon \vdash^{\gamma_2}_{R_1} \mathbf{rec}\ f(x).\ {e_3}^0 : B \to A$
- $\Psi_3 \mid \Theta_3 \vdash^{\gamma_2} E_1$ **wf**
- $\Psi_3 \vdash^{\gamma_1} B \to A$ **wf**

for some γ_2. By inverting the typing of $\mathbf{rec}\ f(x).\ {e_3}^0$, we have $\Psi_3 \mid^{\gamma_2} f{:}^{\gamma_3} B \to A, x{:}^{\gamma_4} B \vdash^{\gamma_4}_{R_1} {e_3}^0 : A$ for some γ_3 and γ_4. We take $g = \mathbf{gensym}\,(\mathbf{Dom}_V(\Psi_3))$ and $y = \mathbf{gensym}\,(\mathbf{Dom}_V(\Psi_3), g)$. Let $R_2 = R_1, f \coloneqq g, x \coloneqq y$. As $g \notin \mathbf{Dom}_V(\Psi_3)$ and $y \notin \mathbf{Dom}_V(\Psi_3)$, we apply Corollary 3 to get $\Psi_3, g : B \to A\ @\ \gamma_3 :\succeq \gamma_2, y : B\ @\ \gamma_4 :\succeq \gamma_3 \mid^{\gamma_4} \varepsilon \vdash^{\gamma_4}_{R_2} {e_3}^0 : A$. Let $\Psi_4 = \Psi_3, g : B \to A\ @\ \gamma_3 :\succeq \gamma_2, y : B\ @\ \gamma_4 :\succeq \gamma_3$. From $\Psi_3 \mid \Theta_3 \vdash^{\gamma_2} E_1$ **wf** and $\Psi_2 \mid \Theta_2 \vdash^{\gamma_1} v_1 : B \to A$ **val**, we have $\Psi_4 \mid \Theta_3 \vdash^{\gamma_3} E_1, g \coloneqq v_1$ **wf** by WF-VEnv-Var, weakening along $\Psi_2 \subseteq \Psi_3$, $\Psi_3 \subseteq \Psi_4$, and $\Theta_2 \subseteq \Theta_3$, and Lemma 27. Then, as we have $\Psi_3 \mid \Theta_3 \vdash^{\gamma_1} v_2 : B$ **val**, we have $\Psi_4 \mid \Theta_3 \vdash^{\gamma_4} E_1, g \coloneqq v_1, y \coloneqq v_2$ **wf** by WF-VEnv-Var, weakening along $\Psi_3 \subseteq \Psi_4$, and Lemma 27. We also have $\Psi_4 \vdash S_3 \colon \Theta_3$ by weakening along $\Psi_3 \subseteq \Psi_4$. Let $E_2 = E_1, g \coloneqq v_1, y \coloneqq v_2$. Hence, we can apply IH on ${e_3}^0$ to get either of the following.

- $\mathcal{V}^{\mathrm{rt}}({e_3}^0, E_2, R_2, \mathbf{Dom}_V(\Psi_4), S_3, k_2 - 1) = \mathbf{Timeout}$
- We have v_3, Ψ_5, S_4 and Θ_4 such that
  - $\mathcal{V}^{\mathrm{rt}}({e_3}^0, E_2, R_2, \mathbf{Dom}_V(\Psi_4), S_3, k_2 - 1) = \mathbf{Done}\,(v_3, \mathbf{Dom}_V(\Psi_5), S_4)$
  - $\Psi_5 \mid \Theta_4 \vdash^{\gamma_4} v_3 : A$ **val**
  - $\Psi_5 \vdash S_4 \colon \Theta_4$
  - $\Psi_4 \subseteq \Psi_5$
  - $\Theta_3 \subseteq \Theta_4$

Here, $\mathcal{V}^{\mathrm{rt}}(e^0, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2) = \mathbf{Done}\,(v_3, \mathbf{Dom}_V(\Psi_5), S_4)$ from the definition of $\mathcal{V}^{\mathrm{rt}}$. Hence, it suffices to show that $\Psi_5 \mid \Theta_4 \vdash^{\gamma_1} v_3 : A$ **val**, which can be shown by Lemma 27.

Case $e^0 = \langle {e_1}^1 \rangle$: By Lemma 37, it is derived by R-T-Quote or R-T-Polycls. After repeating the inversion, we have

$$\Psi_1 \mid^{\delta_0} [\varrho_1]^{\delta_1}, \ldots, [\varrho_k]^{\delta_k}, \blacktriangleright^{\gamma_2} \vdash^{\delta_k, \gamma_2}_{R} {e_1}^1 : A_1.$$

for some k, $\delta_1, \ldots, \delta_k$, $\varrho_1, \ldots, \varrho_k$, A_1 and γ_2 such that $k \geq 0$, $\delta_0 = \gamma_1$, and $A = \forall \varrho_1 \ldots \forall \varrho_k . \langle A_1 \rangle^{\gamma_2}$.

By repeating Corollary 3, we have

$$\Psi_1, [\delta_1 :\succeq \delta_0, \varrho_1], \ldots, [\delta_k :\succeq \delta_{k-1}, \varrho_k] \mid^{\delta_k, \gamma_2} \varepsilon \vdash^{\delta_k, \gamma_2}_{R} {e_1}^1 : A_1.$$

Let $\Psi_2 = \Psi_1, [\delta_1 :\succeq \delta_0, \varrho_1], \ldots, [\delta_k :\succeq \delta_{k-1}, \varrho_k]$. We have $\mathbf{Dom}_V(\Psi_2) = \mathbf{Dom}_V(\Psi_1)$ and $\Psi_2 \mid \Theta_1 \vdash^{\delta_k} E$ **wf** by weakening and repeated application of WF-VEnv-Polycls. Also, $\Psi_2 \vdash S_1 \colon \Theta_1$ follows by weakening. Hence, we apply the induction hypothesis on ${e_1}^1$ and get either of the following.

- $\mathcal{V}^{\mathrm{fut}}(0, {e_1}^1, E, R, \mathbf{Dom}_V(\Psi_2), S_1, k_2 - 1) = \mathbf{Timeout}$
- We have ${e_2}^0$, Ψ_3, S_2 and Θ_2 such that
  - $\mathcal{V}^{\mathrm{fut}}(0, {e_1}^1, E, R, \mathbf{Dom}_V(\Psi_2), S_1, k_2 - 1) = \mathbf{Done}\,({e_2}^0, \mathbf{Dom}_V(\Psi_3), S_2)$

- $\Psi_3 \mid^{\gamma_2} \varepsilon \vdash^{\gamma_2}_{\mathbf{id}} {e_2}^0 : A_1$
- $\Psi_3 \vdash S_2 : \Theta_2$
- $\Psi_2 \subseteq \Psi_3$
- $\Theta_1 \subseteq \Theta_2$

The first case is trivial, and we consider the second case.
Since $\mathbf{Dom}_V(\Psi_2) = \mathbf{Dom}_V(\Psi_1)$, the definition of $\mathcal{V}^{\mathrm{rt}}$ gives

$$\mathcal{V}^{\mathrm{rt}}(\langle {e_1}^1 \rangle, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2) = \mathbf{Done}\,(\langle {e_2}^0 \rangle, \mathbf{Dom}_V(\Psi_3), S_2).$$

It remains to show $\Psi_3 \mid \Theta_2 \vdash^{\gamma_1} \langle {e_2}^0 \rangle : A$ **val**. By repeated demotion, using $\Psi_2 \subseteq \Psi_3$, we have

$$\Psi_3 \mid^{\delta_0} [\varrho_1]^{\delta_1}, \ldots, [\varrho_k]^{\delta_k}, \blacktriangleright^{\gamma_2} \vdash^{\delta_k, \gamma_2}_{\mathbf{id}} {e_2}^0 : A_1.$$

Then applying R-T-Quote and R-T-Polycls repeatedly, we have

$$\Psi_3 \mid^{\delta_0} \varepsilon \vdash^{\gamma_1}_{\mathbf{id}} \langle {e_2}^0 \rangle : A.$$

Hence, we can apply VT-Code to get the desired value typing judgment.

Case $e^0 = \$_0\{{e_1}^0\}$**:** By Lemma 37, it is derived by R-T-Appcls or R-T-Splice. The former case is already discussed, and we consider the latter case. We are given

$$\textsc{R-T-Splice}\ \frac{\Psi_1 \mid^{\gamma_1} \blacktriangleleft^{\gamma_2}_0 \vdash^{\gamma_2}_R {e_1}^0 : \langle A \rangle^{\gamma_3} \qquad \Psi_1 \mid^{\gamma_1} \varepsilon \vdash \gamma_3 \preceq \gamma_1}{\Psi_1 \mid^{\gamma_1} \varepsilon \vdash^{\gamma_1}_R \$_0\{{e_1}^0\} : A}$$

for some γ_2 and γ_3.
Let $\Psi_2 = \Psi_1, \gamma_2(:\succeq \gamma_1 / :\supseteq \gamma_1)$. By Corollary 3, we have $\Psi_2 \mid^{\gamma_2} \varepsilon \vdash^{\gamma_2}_R {e_1}^0 : \langle A \rangle^{\gamma_3}$. We have $\mathbf{Dom}_V(\Psi_2) = \mathbf{Dom}_V(\Psi_1)$, and $\Psi_2 \mid \Theta_1 \vdash^{\gamma_2} E$ **wf** by WF-VEnv-Shut. Also, $\Psi_2 \vdash S_1 : \Theta_1$ follows by weakening. Hence, by the induction hypothesis on ${e_1}^0$, we have either of the following.

- $\mathcal{V}^{\mathrm{rt}}({e_1}^0, E, R, \mathbf{Dom}_V(\Psi_2), S_1, k_2 - 1) = \mathbf{Timeout}$
- We have v_1, Ψ_3, S_2 and Θ_2 such that
  - $\mathcal{V}^{\mathrm{rt}}({e_1}^0, E, R, \mathbf{Dom}_V(\Psi_2), S_1, k_2 - 1) = \mathbf{Done}\,(v_1, \mathbf{Dom}_V(\Psi_3), S_2)$
  - $\Psi_3 \mid \Theta_2 \vdash^{\gamma_2} v_1 : \langle A \rangle^{\gamma_3}$ **val**
  - $\Psi_3 \vdash S_2 : \Theta_2$
  - $\Psi_2 \subseteq \Psi_3$
  - $\Theta_1 \subseteq \Theta_2$

The first case is trivial, and we consider the second case. By inverting the typing of v_1, we have $\Psi_3 \mid^{\gamma_2} \varepsilon \vdash^{\gamma_2}_{\mathbf{id}} \langle {e_2}^0 \rangle : \langle A \rangle^{\gamma_3}$ for some ${e_2}^0$ such that $v_1 = \langle {e_2}^0 \rangle$. By inversion of R-T-Quote, we obtain $\Psi_3 \mid^{\gamma_2} \blacktriangleright^{\gamma_3} \vdash^{\gamma_2, \gamma_3}_{\mathbf{id}} {e_2}^0 : A$. Hence Lemma 17 gives $\Psi_3 \mid^{\gamma_3} \varepsilon \vdash^{\gamma_3}_{\mathbf{id}} {e_2}^0 : A$. By weakening the premise $\Psi_1 \mid^{\gamma_1} \varepsilon \vdash \gamma_3 \preceq \gamma_1$ along $\Psi_1 \subseteq \Psi_2$ and $\Psi_2 \subseteq \Psi_3$, and then applying Lemma 5, we obtain $\Psi_3 \vdash \gamma_3 \preceq \gamma_1$. Hence Lemma 13 gives $\Psi_3 \mid^{\gamma_1} \varepsilon \vdash^{\gamma_1}_{\mathbf{id}} {e_2}^0 : A$. Moreover, $\Psi_3 \mid \Theta_2 \vdash^{\gamma_1} E$ **wf** follows by weakening, since $\Psi_1 \subseteq \Psi_2$, $\Psi_2 \subseteq \Psi_3$, and $\Theta_1 \subseteq \Theta_2$. Thus, by the induction hypothesis on ${e_2}^0$, we have the following.

- $\mathcal{V}^{\mathrm{rt}}({e_2}^0, E, \mathbf{id}, \mathbf{Dom}_V(\Psi_3), S_2, k_2 - 1) = \mathbf{Timeout}$
- We have v_2, Ψ_4, S_3 and Θ_3 such that
  - $\mathcal{V}^{\mathrm{rt}}({e_2}^0, E, \mathbf{id}, \mathbf{Dom}_V(\Psi_3), S_2, k_2 - 1) = \mathbf{Done}\,(v_2, \mathbf{Dom}_V(\Psi_4), S_3)$
  - $\Psi_4 \mid \Theta_3 \vdash^{\gamma_1} v_2 : A$ **val**
  - $\Psi_4 \vdash S_3 : \Theta_3$
  - $\Psi_3 \subseteq \Psi_4$
  - $\Theta_2 \subseteq \Theta_3$

The first case is trivial, and we consider the second case. Since $\mathbf{Dom}_V(\Psi_2) = \mathbf{Dom}_V(\Psi_1)$, the definition of $\mathcal{V}^{\mathrm{rt}}$ gives $\mathcal{V}^{\mathrm{rt}}(\$_0\{{e_1}^0\}, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2) = \mathbf{Done}\,(v_2, \mathbf{Dom}_V(\Psi_4), S_3)$. Therefore we are done.

Case $e^0 = \mathbf{ref}\,({e_1}^0)$**:** We are given

$$\text{R-T-Ref}\ \frac{\Psi_1 \mid^{\gamma_1} \varepsilon \vdash^{\gamma_1}_{R} {e_1}^0 : A_1}{\Psi_1 \mid^{\gamma_1} \varepsilon \vdash^{\gamma_1}_{R} \mathbf{ref}\,({e_1}^0) : A_1\ \mathbf{ref}}$$

for some A_1 where $A = A_1\ \mathbf{ref}$. By the induction hypothesis on ${e_1}^0$, we have either of the following.

- $\mathcal{V}^{\mathrm{rt}}({e_1}^0, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2 - 1) = \mathbf{Timeout}$
- We have v_1, Ψ_2, S_2 and Θ_2 such that
  - $\mathcal{V}^{\mathrm{rt}}({e_1}^0, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2 - 1) = \mathbf{Done}\,(v_1, \mathbf{Dom}_V(\Psi_2), S_2)$
  - $\Psi_2 \mid \Theta_2 \vdash^{\gamma_1} v_1 : A_1\ \mathbf{val}$
  - $\Psi_2 \vdash S_2 \colon \Theta_2$
  - $\Psi_1 \subseteq \Psi_2$
  - $\Theta_1 \subseteq \Theta_2$

The first case is trivial, and we consider the second case. Let l be a location such that $\mathbf{alloc}\,(S_2) = \mathbf{Done}\,(l)$. Since $l \notin \mathbf{Dom}_L(S_2)$, define $S_3 = S_2, l \coloneqq v_1$ and $\Theta_3 = \Theta_2, l : A_1$. By the definition of $\mathcal{V}^{\mathrm{rt}}$,

$$\mathcal{V}^{\mathrm{rt}}(\mathbf{ref}\,({e_1}^0), E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2) = \mathbf{Done}\,(l, \mathbf{Dom}_V(\Psi_2), S_3).$$

It remains to show $\Psi_2 \mid \Theta_3 \vdash^{\gamma_1} l : A_1\ \mathbf{ref}\ \mathbf{val}$ and $\Psi_2 \vdash S_3 \colon \Theta_3$. The former follows from VT-Loc. The latter follows from T-Store, using the store typing of S_2 and the value typing of v_1.

Case $e^0 = {e_1}^0 \coloneqq {e_2}^0$**:** By Lemma 40.

Case $e^0 = \mathbf{deref}({e_1}^0)$**:** By Lemma 39.

Item 2 : The case where $k_2 = 0$ is trivial. We consider the case where $k_2 > 0$. We perform case analysis on e and its last derivation.

Case Last derivation is R-T-Appcls**:** Let $\varrho = \delta_1 :\succeq \delta'_1, \ldots, \delta_k :\succeq \delta'_k$ and $\sigma = \delta_1 \coloneqq \delta''_1, \ldots, \delta_k \coloneqq \delta''_k$. We are given

$$\text{R-T-Appcls}\ \frac{\begin{array}{c}\Psi_1 \mid^{\gamma_0,\ldots,\gamma_{k_1+}} \varepsilon \vdash^{\gamma_0,\ldots,\gamma_{k_1+}}_{R} {e_1}^{{k_1}^+} : \forall\varrho.A' \\ \Psi_1 \mid^{\gamma_0,\ldots,\gamma_{k_1+}} \varepsilon \vdash \delta'_i \preceq \delta''_i \text{ for } 1 \le i \le k \qquad \Psi_1 \mid^{\gamma_0,\ldots,\gamma_{k_1+}} \varepsilon \vdash \delta''_{i+1} \subseteq \delta''_i \text{ for } 0 \le i < k\end{array}}{\Psi_1 \mid^{\gamma_0,\ldots,\gamma_{k_1+}} \varepsilon \vdash^{\gamma_0,\ldots,\gamma_{k_1+}}_{R} {e_1}^{{k_1}^+} : A'[\sigma]}$$

where $\delta''_0 = \gamma_{k_1+}$ and $A = A'[\sigma]$.

By the induction hypothesis applied to the premise typing derivation, either of the following holds:

- $\mathcal{V}^{\mathrm{fut}}(k_1, {e_1}^{{k_1}^+}, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2) = \mathbf{Timeout}$
- We have ${e_2}^{k_1}$, Ψ_2, S_2 and Θ_2 such that
  - $\mathcal{V}^{\mathrm{fut}}(k_1, {e_1}^{{k_1}^+}, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2) = \mathbf{Done}\,({e_2}^{k_1}, \mathbf{Dom}_V(\Psi_2), S_2)$
  - $\Psi_2 \mid^{\gamma_1,\ldots,\gamma_{k_1}} \varepsilon \vdash^{\gamma_1,\ldots,\gamma_{k_1}}_{\mathbf{id}} {e_2}^{k_1} : \forall\varrho.A'$
  - $\Psi_2 \vdash S_2 \colon \Theta_2$
  - $\Psi_1 \subseteq \Psi_2$
  - $\Theta_1 \subseteq \Theta_2$

The first case is trivial, and we consider the second case. By weakening along $\Psi_1 \subseteq \Psi_2$ and then applying Lemma 5, the reachability/visibility premises become global judgments over Ψ_2. Applying Lemma 6 to the residual stack $\gamma_1, \ldots, \gamma_k$ gives

- $\Psi_2 \mid^{\gamma_1,\ldots,\gamma_{k_1}} \varepsilon \vdash \delta'_i \preceq \delta''_i$ for each i with $1 \le i \le k$,
- $\Psi_2 \mid^{\gamma_1,\ldots,\gamma_{k_1}} \varepsilon \vdash \delta''_{i+1} \subseteq \delta''_i$ for each i with $0 \le i < k$.

Reapplying R-T-Appcls gives

$$\Psi_2 \mid^{\gamma_1,\ldots,\gamma_{k_1}} \varepsilon \vdash^{\gamma_1,\ldots,\gamma_{k_1}}_{\mathbf{id}} {e_2}^{k_1} : A'[\sigma].$$

This is what we want.

Case $e^{{k_1}^+} = \lambda x.\ {e_1}^{{k_1}^+}$: By Lemma 37, it is derived by R-T-Func or R-T-Polycls.

In the former case, we are given

$$\textsc{R-T-Func}\ \frac{\begin{array}{c}\Psi_1 \mid^{\gamma_0,\dots,\gamma_{{k_1}^+}} x{:}^{\gamma'_1} A_1 \vdash_R^{\gamma_0,\dots,\gamma_{k_1},\gamma'_1} {e_1}^{{k_1}^+} : A_2 \\ \gamma'_1 \notin \mathbf{FC}(A_2) \qquad \Psi_1 \mid^{\gamma_0,\dots,\gamma_{{k_1}^+}} \varepsilon \vdash^{\gamma_0,\dots,\gamma_{k_1},\gamma_{{k_1}^+}} \mathbf{wf}\end{array}}{\Psi_1 \mid^{\gamma_0,\dots,\gamma_{{k_1}^+}} \varepsilon \vdash_R^{\gamma_0,\dots,\gamma_{{k_1}^+}} \lambda x.\ {e_1}^{{k_1}^+} : A_1 \to A_2}$$

for some A_1, A_2 and γ'_1.

Let $y = \mathbf{gensym}\,(\mathbf{Dom}_V(\Psi_1))$ and $\Psi_2 = \Psi_1, y : A_1 \mathbin{@} \gamma'_1 :\succeq \gamma_{{k_1}^+}$. By Corollary 3, we have

$$\Psi_2 \mid^{\gamma_0,\dots,\gamma_{k_1},\gamma'_1} \varepsilon \vdash_{R,x:=y}^{\gamma_0,\dots,\gamma_{k_1},\gamma'_1} {e_1}^{{k_1}^+} : A_2.$$

Moreover, $\mathbf{Dom}_V(\Psi_2) = \mathbf{Dom}_V(\Psi_1), y$. By weakening along $\Psi_1 \subseteq \Psi_2$, we also have $\Psi_2 \mid \Theta_1 \vdash^{\gamma_0} E\ \mathbf{wf}$ and $\Psi_2 \vdash S_1 : \Theta_1$. Hence, by the induction hypothesis on ${e_1}^{{k_1}^+}$, either of the following holds:

- $\mathcal{V}^{\mathrm{fut}}(k_1, {e_1}^{{k_1}^+}, E, (R, x \coloneqq y), \mathbf{Dom}_V(\Psi_2), S_1, k_2 - 1) = \mathbf{Timeout}$
- We have ${e_2}^{k_1}$, Ψ_3, S_2, and Θ_2 such that
  - $\mathcal{V}^{\mathrm{fut}}(k_1, {e_1}^{{k_1}^+}, E, (R, x \coloneqq y), \mathbf{Dom}_V(\Psi_2), S_1, k_2 - 1) = \mathbf{Done}\,({e_2}^{k_1}, \mathbf{Dom}_V(\Psi_3), S_2)$
  - $\Psi_3 \mid^{\gamma_1,\dots,\gamma_{k_1},\gamma'_1} \varepsilon \vdash_{\mathbf{id}}^{\gamma_1,\dots,\gamma_{k_1},\gamma'_1} {e_2}^{k_1} : A_2$
  - $\Psi_3 \vdash S_2 : \Theta_2$
  - $\Psi_2 \subseteq \Psi_3$
  - $\Theta_1 \subseteq \Theta_2$

The first case is trivial, and we consider the second case. Since $\gamma'_1 \notin \mathbf{FC}(A_2)$, the demotion does not change the result type: $A_2[\gamma'_1 \coloneqq \gamma'_2] = A_2$. Thus, by Corollary 4, using $\Psi_2 \subseteq \Psi_3$, we have

$$\Psi_3 \mid^{\gamma_1,\dots,\gamma_{k_1}} y{:}^{\gamma'_2} A_1 \vdash_{\mathbf{id}}^{\gamma_1,\dots,\gamma_{k_1},\gamma'_2} {e_2}^{k_1} : A_2.$$

for fresh γ'_2. Reapplying R-T-Func gives

$$\Psi_3 \mid^{\gamma_1,\dots,\gamma_{k_1}} \varepsilon \vdash_{\mathbf{id}}^{\gamma_1,\dots,\gamma_{k_1}} \lambda y.\ {e_2}^{k_1} : A_1 \to A_2.$$

Since $\mathcal{V}^{\mathrm{fut}}(k_1, \lambda x.\ {e_1}^{{k_1}^+}, E, R, \mathbf{Dom}_V(\Psi_1), S_1, k_2) = \mathbf{Done}\,(\lambda y.\ {e_2}^{k_1}, \mathbf{Dom}_V(\Psi_3), S_2)$ by the definition of $\mathcal{V}^{\mathrm{fut}}$, this is the desired result.

We next consider the case where $e^{{k_1}^+}$ is derived by R-T-Polycls. In this case, we are given

$$\textsc{R-T-Polycls}\ \frac{\begin{array}{c}\Psi_1 \mid^{\gamma_0,\dots,\gamma_{{k_1}^+}} [\delta_1 :\succeq \delta'_1, \dots, \delta_k :\succeq \delta'_k]^{\delta_0} \vdash_R^{\gamma_0,\dots,\gamma_{k_1},\delta_0} \lambda x.\ {e_1}^{{k_1}^+} : A' \\ \delta_0 \notin \mathbf{FC}(A') \qquad \Psi_1 \mid^{\gamma_0,\dots,\gamma_{{k_1}^+}} \varepsilon \vdash^{\gamma_0,\dots,\gamma_{k_1},\delta'_0} \mathbf{wf}\end{array}}{\Psi_1 \mid^{\gamma_0,\dots,\gamma_{{k_1}^+}} \varepsilon \vdash_R^{\gamma_0,\dots,\gamma_{k_1},\delta'_0} \lambda x.\ {e_1}^{{k_1}^+} : \forall \delta_1 :\succeq \delta'_1, \dots, \delta_k :\succeq \delta'_k . A'}$$

for some k, $\delta_0, \dots, \delta_k$, $\delta'_0, \dots, \delta'_k$, and A', where $\delta'_0 = \gamma_{{k_1}^+}$ and $A = \forall \delta_1 :\succeq \delta'_1, \dots, \delta_k :\succeq \delta'_k . A'$.

By Corollary 3, the premise typing judgment can be promoted to the reserved-name context. Let

$$\Psi_2 = \Psi_1, [\delta_0 :\succeq \delta'_0, \dots, \delta_k :\succeq \delta'_k].$$

Then

$$\Psi_2 \mid^{\gamma_0,\dots,\gamma_{k_1},\delta_0} \varepsilon \vdash_R^{\gamma_0,\dots,\gamma_{k_1},\delta_0} \lambda x.\ {e_1}^{{k_1}^+} : A'.$$

Moreover, $\mathbf{Dom}_V(\Psi_2) = \mathbf{Dom}_V(\Psi_1)$, and the environment and store typing premises are preserved by weakening along $\Psi_1 \subseteq \Psi_2$. Hence, by the induction hypothesis applied to this promoted derivation, either of the following holds:

- $\mathcal{V}^{\mathrm{fut}}(k_1, \lambda x.\ {e_1}^{{k_1}^+}, E, R, \mathbf{Dom}_V(\Psi_2), S_1, k_2) = \mathbf{Timeout}$
- We have ${e_2}^{k_1}$, Ψ_3, S_2, and Θ_2 such that
  - $\mathcal{V}^{\mathrm{fut}}(k_1, \lambda x.\ {e_1}^{{k_1}^+}, E, R, \mathbf{Dom}_V(\Psi_2), S_1, k_2) = \mathbf{Done}\,({e_2}^{k_1}, \mathbf{Dom}_V(\Psi_3), S_2)$

- $\Psi_3 \mid^{\gamma_1,\dots,\gamma_{k_1},\delta_0} \varepsilon \vdash_{\mathbf{id}}^{\gamma_1,\dots,\gamma_{k_1},\delta_0} {e_2}^{k_1} : A'$
- $\Psi_3 \vdash S_2 \colon \Theta_2$
- $\Psi_2 \subseteq \Psi_3$
- $\Theta_1 \subseteq \Theta_2$

The first case is trivial, and we consider the second case. By the definition of $\mathcal{V}^{\mathrm{fut}}$ for functions, there are some y and ${e_2'}^{k_1}$ such that ${e_2}^{k_1} = \lambda y.\ {e_2'}^{k_1}$. Hence ${e_2}^{k_1}$ is an introduction form. By applying the polymorphic-classifier case of Corollary 4, using $\Psi_2 \subseteq \Psi_3$, we obtain the local premise

$$\Psi_3 \mid^{\gamma_1,\dots,\gamma_{k_1},\delta_0'} [\delta_1 :\succeq \delta_1', \dots, \delta_k :\succeq \delta_k']^{\delta_0} \vdash_{\mathbf{id}}^{\gamma_1,\dots,\gamma_{k_1},\delta_0} {e_2}^{k_1} : A'.$$

Reapplying R-T-Polycls gives

$$\Psi_3 \mid^{\gamma_1,\dots,\gamma_{k_1},\delta_0'} \varepsilon \vdash_{\mathbf{id}}^{\gamma_1,\dots,\gamma_{k_1},\delta_0'} {e_2}^{k_1} : \forall \delta_1 :\succeq \delta_1', \dots, \delta_k :\succeq \delta_k'. A'.$$

Since $\mathbf{Dom}_V(\Psi_2) = \mathbf{Dom}_V(\Psi_1)$ and $\delta_0' = \gamma_{{k_1}^+}$, this is exactly the typing required for the result of the original $\mathcal{V}^{\mathrm{fut}}$ computation.

Case $e^{{k_1}^+} = \mathbf{rec}\ f(x).\ {e_1}^{{k_1}^+}$**:** Mostly the same as the previous case, but we need to use R-T-Rec instead of R-T-Func.

Case $e^{{k_1}^+} = \langle {e_1}^{k_1+2} \rangle$**:** By Corollary 4.

Case $e^{{k_1}^+} = \$_{k_3}\{{e_1}^{{k_1}^+ - k_3}\}$ where ${k_1}^+ > k_3$**:** By Corollary 4.

Case $e^{{k_1}^+} = \$_{{k_1}^+}\{{e_1}^0\}$**:** By Lemma 37, it is derived by R-T-Appcls or R-T-Splice. The former case is already discussed, and we consider the latter case.

We are given

$$\textsc{R-T-Splice}\ \frac{\Psi_1 \mid^{\gamma_0,\dots,\gamma_{{k_1}^+}} \blacktriangleleft^{\delta_1}_{{k_1}^+} \vdash_R^{\delta_1} {e_1}^0 : \langle A \rangle^{\delta_2} \qquad \Psi_1 \mid^{\gamma_0,\dots,\gamma_{{k_1}^+}} \varepsilon \vdash \delta_2 \preceq \gamma_{{k_1}^+} \qquad \delta_1 \notin \mathbf{FC}(A)}{\Psi_1 \mid^{\gamma_0,\dots,\gamma_{{k_1}^+}} \varepsilon \vdash_R^{\gamma_0,\dots,\gamma_{{k_1}^+}} \$_{{k_1}^+}\{{e_1}^0\} : A}$$

for some δ_1 and δ_2. Let $\Psi_2 = \Psi_1, \delta_1(:\succeq \gamma_0 / :\supseteq \gamma_{{k_1}^+})$. By Corollary 3, we have

$$\Psi_2 \mid^{\delta_1} \varepsilon \vdash_R^{\delta_1} {e_1}^0 : \langle A \rangle^{\delta_2}.$$

Moreover, $\mathbf{Dom}_V(\Psi_2) = \mathbf{Dom}_V(\Psi_1)$. By weakening along $\Psi_1 \subseteq \Psi_2$, we have $\Psi_2 \mid \Theta_1 \vdash^{\delta_1} E\ \mathbf{wf}$ by WF-VEnv-Shut, and $\Psi_2 \vdash S_1 \colon \Theta_1$. Hence, by the induction hypothesis on ${e_1}^0$, either of the following holds:

- $\mathcal{V}^{\mathrm{rt}}({e_1}^0, E, R, \mathbf{Dom}_V(\Psi_2), S_1, k_2 - 1) = \mathbf{Timeout}$
- We have v_1, Ψ_3, S_2 and Θ_2 such that
  - $\mathcal{V}^{\mathrm{rt}}({e_1}^0, E, R, \mathbf{Dom}_V(\Psi_2), S_1, k_2 - 1) = \mathbf{Done}\,(v_1, \mathbf{Dom}_V(\Psi_3), S_2)$
  - $\Psi_3 \mid \Theta_2 \vdash^{\delta_1} v_1 : \langle A \rangle^{\delta_2}\ \mathbf{val}$
  - $\Psi_3 \vdash S_2 \colon \Theta_2$
  - $\Psi_2 \subseteq \Psi_3$
  - $\Theta_1 \subseteq \Theta_2$

The first case is trivial, and we consider the second case. By inverting the typing of v_1 and applying Lemma 17, we have $\Psi_3 \mid^{\delta_2} \varepsilon \vdash_{\mathbf{id}}^{\delta_2} {e_2}^0 : A$ for some ${e_2}^0$ such that $v_1 = \langle {e_2}^0 \rangle$. By weakening the premise $\Psi_1 \mid^{\gamma_0,\dots,\gamma_{{k_1}^+}} \varepsilon \vdash \delta_2 \preceq \gamma_{{k_1}^+}$ along $\Psi_1 \subseteq \Psi_2$ and $\Psi_2 \subseteq \Psi_3$, and then applying Lemma 5, we obtain $\Psi_3 \vdash \delta_2 \preceq \gamma_{{k_1}^+}$. Hence Lemma 13 gives

$$\Psi_3 \mid^{\gamma_1,\dots,\gamma_{k_1}} \varepsilon \vdash_{\mathbf{id}}^{\gamma_1,\dots,\gamma_{k_1}} {e_2}^0 : A.$$

Since $\mathbf{Dom}_V(\Psi_2) = \mathbf{Dom}_V(\Psi_1)$, the definition of $\mathcal{V}^{\mathrm{fut}}$ gives the desired result.

$\square$

Corollary 5 (Soundness of Surface Typing)**.** *If $\varepsilon \vdash^{!} e^0 : A$, then either of the following holds.*

- $\mathcal{V}^{\mathrm{rt}}(e^0, \varepsilon, \mathbf{id}, \varepsilon, \varepsilon, k_2) = \mathbf{Timeout}$
- $\mathcal{V}^{\mathrm{rt}}(e^0, \varepsilon, \mathbf{id}, \varepsilon, \varepsilon, k_2) = \mathbf{Done}\,(v, N, S)$ *where* $\Psi \mid \Theta \vdash^{!} v : A\ \mathbf{val}$, $\mathbf{Dom}_V(\Psi) = N$ *and* $\Psi \vdash S \colon \Theta$ *for some* Ψ *and* Θ.

3.10 Safety of Offline Code Generation

Lemma 41 (Discarding Reserved Names)**.** *If* $\Psi \mid^{!} \varepsilon \vdash^{!}_{\mathbf{id}} e^0 : A$*, then* $\varepsilon \mid^{!} \varepsilon \vdash^{!}_{\mathbf{id}} e^0 : A$*.*

Proof. We prove the following generalized statements by mutual induction on the derivations.

1. If $\Psi \mid^{!} \Gamma \vdash^{\vec{\gamma}^+} \mathbf{wf}$, then $\varepsilon \mid^{!} \Gamma \vdash^{\vec{\gamma}^+} \mathbf{wf}$.
2. If $\Psi \mid^{!} \Gamma \vdash^{\vec{\gamma}^+} A\ \mathbf{wf}$, then $\varepsilon \mid^{!} \Gamma \vdash^{\vec{\gamma}^+} A\ \mathbf{wf}$.
3. If $\Psi \mid^{!} \Gamma \vdash \gamma_1 \preceq \gamma_2$ where $\gamma_1 \in \mathbf{Dom}_C(\Gamma)$ and $\gamma_2 \in \mathbf{Dom}_C(\Gamma)$, then $\varepsilon \mid^{!} \Gamma \vdash \gamma_1 \preceq \gamma_2$.
4. If $\Psi \mid^{!} \Gamma \vdash \gamma_1 \subseteq \gamma_2$ where $\gamma_1 \in \mathbf{Dom}_C(\Gamma)$ and $\gamma_2 \in \mathbf{Dom}_C(\Gamma)$, then $\varepsilon \mid^{!} \Gamma \vdash \gamma_1 \subseteq \gamma_2$.
5. If $\Psi \mid^{!} \Gamma \vdash^{\vec{\gamma_1}^+}_{\mathbf{id}} e : A$, then $\varepsilon \mid^{!} \Gamma \vdash^{\vec{\gamma_1}^+}_{\mathbf{id}} e : A$.

Then, the main statement is the special case of Item 5, where $\Gamma = \varepsilon$. □

Theorem 2 (Safety of Offline Code Generation)**.** *If* $\varepsilon \vdash^{!} {e_1}^0 : \langle A \rangle^{!}$ *and* $\mathcal{V}^{\mathrm{rt}}({e_1}^0, \varepsilon, \mathbf{id}, \varepsilon, \varepsilon, k_2) = \mathbf{Done}\,(v, N, S)$*, then* $v = \langle {e_2}^0 \rangle$ *and* $\varepsilon \vdash^{!} {e_2}^0 : A$ *for some* ${e_2}^0$*.*

Proof. From Theorem 1, we have some Ψ, Θ such that $\Psi \mid \Theta \vdash^{!} v : \langle A \rangle^{!}\ \mathbf{val}$ where $\mathbf{Dom}_V(\Psi) = N$ and $\mathbf{Dom}_L(\Theta) = \mathbf{Dom}_L(S)$. By inversion of VT-Code, there is some e_2 such that $v = \langle e_2 \rangle$ and

$$\Psi \mid^{!} \varepsilon \vdash^{!}_{\mathbf{id}} \langle e_2 \rangle : \langle A \rangle^{!}.$$

By inversion of R-T-Quote, we have

$$\Psi \mid^{!} \blacktriangleright^{!} \vdash^{!,!}_{\mathbf{id}} e_2 : A.$$

Hence Lemma 17 gives $\Psi \mid^{!} \varepsilon \vdash^{!}_{\mathbf{id}} e_2 : A$. Finally, Lemma 41 yields $\varepsilon \mid^{!} \varepsilon \vdash^{!}_{\mathbf{id}} e_2 : A$. □

4 Counterexamples for Type Soundness of $\lambda^{[]}$

$\lambda^{[]}$ [1] is an attempt to design a contextual-types-based type system for MetaML-style MSP with mutable state. The idea is to annotate code types with contexts, which is similar in spirit to our use of RECs. However, contextual types in $\lambda^{[]}$ account only for what we call reachability and cannot capture visibility. As a result, $\lambda^{[]}$ admits well-typed programs that nevertheless cause scope extrusion. In fact, the scope-extrusion example from Section 2.3 is typable in $\lambda^{[]}$. Below, we present the corresponding program in $\lambda^{[]}$.

$$
\begin{aligned}
;\vdash\ & \mathtt{let}\ r^{[](\mathtt{int})\ \mathtt{ref}} = \mathtt{ref}_{\{[](\mathtt{int})\}}(\uparrow 1)\ \mathtt{in} \\
& \uparrow\uparrow (\lambda x^{\mathtt{int}}. \downarrow\downarrow (\mathtt{set}_{\{[](\mathtt{int})\}}(r, \uparrow (\mathtt{let}\ y^{\mathtt{int}}\ \mathtt{=}\ \uparrow (x)\ \mathtt{in}\ 1));\ \uparrow\uparrow x)); \\
& \mathtt{get}_{\{\mathtt{int} \to \mathtt{int}\}}(r) \\
:\ & \mathtt{int} \to \mathtt{int}
\end{aligned}
$$

The semantics of $\lambda^{[]}$ uses a local store, which does not allow code fragments to escape their defining scopes. Hence, $\mathtt{set}_{\{[](\mathtt{int})\}}$ on the second line gets stuck, because the code fragment $\uparrow (\mathtt{let}\ y^{\mathtt{int}}\ \mathtt{=}\ \uparrow (x)\ \mathtt{in}\ 1)$ cannot escape the scope of x. This is a counterexample to progress for $\lambda^{[]}$.

References

[1] Morten Rhiger. Staged computation with staged lexical scope. In Helmut Seidl, editor, *Programming Languages and Systems*, pages 559–578, Tallinn, Estonia, 2012. Springer.